J.B. LOCK

DYNAMICS FOR BEGINNERS

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PREFACE.

THIS work has been written in the hope of supplying a want which I believe is very widely felt; viz. of a book which explains the elementary principles of Dynamics, illustrating them by numerous easy *numerical* examples in a manner suitable for use in Schools with boys of ordinary mathematical attainments.

I have adopted a suggestion,—due I believe to Mr Hayward, F.R.S., of Harrow School—that the first part of the book should treat exclusively of Linear Dynamics; thus avoiding, at the beginning of the subject, all purely geometrical difficulties.

It will be found that, with the exception of one or two Articles, Sections I. and IV. (which may be read separately, and in which the fundamental Principles of Dynamics are explained) demand only a knowledge of Simple Equations in Algebra. In no case is any greater knowledge on the part of the Student assumed, than is denoted by Progressions in Algebra, the Trigonometry of one Angle, and (in Chapters IX. and X.) the Parabola.

I have ventured to suggest *names* for the units of velocity and acceleration, the use of which will be found to simplify considerably the language of the subject.

I should be greatly obliged to those who may make use of the book if they would point out any defects or obscurities in the text or would offer suggestions for its improvement.

In the SECOND EDITION (which has been stereotyped) many verbal alterations have been made, for the sake of greater precision of statement.

The work has been received with such unqualified approval by so many Teachers that no serious alteration has been thought necessary.

Of the value for the purposes of teaching and explanation of the names *velo* and *celo* I have received the very strongest testimony from those best qualified to judge.

In the verbal revision I have received great assistance from Mr J. C. Trautwine, Junior, C.E. of Philadelphia, U.S.A.; who has taken great interest in pointing out to me those passages which might be obscure to Practical Engineers.

In the THIRD EDITION a few corrections have been made and at the request of many teachers a collection of over 200 miscellaneous problems has been added.

J. B. LOCK.

February, 1890.

NOTE.

The Examination Papers appended to the Book in most cases sufficiently indicate the range of the various Examinations.

The following suggestions however may be found useful.

For the Dynamics of the Additional Subjects in the Previous Examination at Cambridge, the Student may confine his attention to Section I. up to the end of page 63, Section II. and Art. 134 of Chapter XI.

For the First M.B. Examination, in addition to the above, the Student should read Section IV.

Those Articles which are marked with an asterisk may in most cases be reserved with advantage for a second reading of the subject.

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SECTION I.

RECTILINEAR MOTION.

CHAPTER I.

VELOCITY, ACCELERATION.

1. The **distance** of a point P from a point A is represented in direction and magnitude by the finite straight line joining A to P.

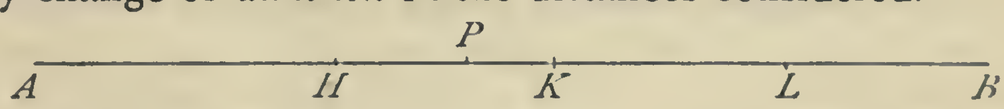
The point P is said to be *in motion relative to* the other point A, when the distance of P from A is changing.

The distance of P from A may be changing in *length*, or it may be changing in *direction*, or it may be changing *both in length and in direction.*

While the *direction* of the distance of P from A remains unchanged the path of P is the straight line AP.

We shall in this section discuss the motion of a point which moves always in a straight line.

Accordingly, for the present, we are not concerned with any change of *direction* in the distances considered.



Draw a straight line AB; let this line extend any length beyond A and beyond B. Let a point P be in motion relative to A in the line AB. Then the distance AP is different at different instants. Thus, suppose the point P at one instant to be at H; at another, at K; at another, at L; H, K and L being points *whose distances from* A *do not change*; then, P being in motion relative to A, P *alters its distance from* A *by the distance* HK in the course of a certain interval of time.

This is what is meant when it is said that P *passes over the distance* HK in a certain interval of time.

We choose **a foot** as our *unit distance.*

2. In what follows we shall frequently use the word **interval** as an abbreviation for *interval of time.*

We choose **a second** as our *unit interval.*

Let the distances from A of the above moving point P be observed at the ends of a series of equal intervals of time. Then if its motion is of such kind that in each of those equal intervals the point P passes over equal distances, and if this is true whatever be the length of these equal intervals, then we say that the point P is moving **uniformly**; hence,

3. DEFINITION. The motion of a point P relative to another point A is said to be **uniform** when it is such that the point P passes over equal distances in equal intervals of time, however short those intervals are.

Let us suppose that the point P is moving uniformly in the straight line AP. By its **velocity** we mean *something* which is doubled, trebled, etc. if the distance which it passes over in a given interval is doubled, trebled, etc., and which is halved, divided by three, etc. if the interval occupied in passing over a given distance is doubled, trebled, etc.; in other words,

4. DEF. The **velocity** of a point which is moving uniformly in a straight line is *that* which varies † directly as the distance passed over in a given interval, and which varies inversely as the interval occupied in passing over a given distance

Velocity is therefore something which we observe in a moving point. It is capable of measurement; for we can say that the velocity of a point is double, or that it is half, or that it is some multiple, of the velocity of another point.

We therefore choose a certain velocity as our *unit* velocity, and give it a name; just as we choose a unit of length and call it a foot; and a unit interval and call it a second...; hence

5. DEF. We choose as our **unit** *velocity* the velocity of a point which moving uniformly passes over the distance 1 foot in the interval 1 second.

† See Lock's Arithmetic, p. 162.

We shall call this *unit velocity* a **velo**.

A *velo* in a given direction, is a velocity of *a foot per second* in that direction.

The word *per* here denotes *for every*.

Example i. A point which moves at the rate of 3 feet per second has a velocity 3 times as great as that of a point which moves at the rate of 1 foot per second. Hence its velocity is 3 *velos*.

Example ii. *Express in* velos *the velocity of a point which moves uniformly at the rate of* 60 miles per hour.

This point in 1 hour passes over 60 miles;

that is, in 60×60 seconds it passes over $60 \times 1760 \times 3$ feet;
but it passes over an equal distance in each second;

therefore in 1 second it passes over $\dfrac{60 \times 1760 \times 3}{60 \times 60}$ feet; that is, 88 feet.

Hence, its velocity is 88 velos.

Example iii. *A point has the uniform* velocity 8 *velos; how many miles does it pass over in an hour?*

The point has 8 velos; therefore, in 1 second it passes over 8 feet; and it passes over equal distances in equal intervals;

therefore in 60×60 seconds it passes over $60 \times 60 \times 8$ feet;

$$\text{that is, } \frac{8 \times 60 \times 60}{1760 \times 3} \text{ miles; or, } 5\tfrac{5}{11} \text{ miles.}$$

NOTE. It is most important to notice that a point which has velocity requires an *interval of time* in which to pass over a *finite* distance.

A point P may have velocity *at* a certain instant, but unless the velocity continues during some interval after that instant, the point does not pass over any finite distance.

For example, a bullet which strikes a target is said to have a certain velocity *at* the instant at which it strikes.

EXAMPLES. I.

N. B. *In each of the following questions the velocity given is uniform.*

1. If I run 100 yards in 11 seconds, what is my velocity in velos?

2. A man runs at the rate of 7 velos; how long would he take to run a mile with this velocity?

3. Express in miles per hour (i) 40 velos, (ii) 100 yds. per minute.

4. How many velos has the extremity (i) of the minute hand of a clock which is 1 foot long, (ii) of the hour hand which is 10 in. long. [*The circumference of a circle* $= \frac{22}{7}$ *of the diameter.*]

5. How many velos has a man standing at the equator, in consequence of the daily revolution of the earth about its axis, the diameter of the earth being, say, 8000 miles?

6. The velocity of sound is 1118 velos; how long does it take to travel 13 miles?

7. Which is the greater velocity, 60 miles an hour or 500 yards in $11\frac{1}{4}$ seconds?

8. Find the ratio of the velocities $2\frac{1}{2}$ miles in 3 minutes and 10 ft. in a quarter of a second.

9. The following are 'records' in foot racing: (i) 100 yds. in 10 secs. (ii) a quarter of a mile in 49 secs. (iii) a mile in 4 min. $12\frac{3}{5}$ secs. (iv) ten miles in 51 min. $6\frac{3}{5}$ secs.; express the average velocity of each in velos.

10. The following are 'records' in bicycle racing: (i) a mile in 2 min. 30 secs. (ii) 20 miles in 59 min. $6\frac{4}{5}$ secs. (iii) 22 miles 150 yds. in 1 hour; express the average velocity of each in velos and in miles per hour.

6. A point which has v velos passes over v feet in 1 second; it therefore passes over (t times v) feet in t seconds.

Hence, if s be the number of feet passed over in t seconds by a point which has a uniform velocity v velos, we have

$$\mathbf{s = vt.}$$

Example. A point passes uniformly over c miles *in* b hours; *express its velocity in* velos.

In $b \times 60 \times 60$ seconds it passes over $c \times 1760 \times 3$ feet, therefore

in 1 second it passes over $\frac{1760 \times 3 \times c}{60 \times 60 \times b}$ feet;

that is, $\frac{22c}{15b}$ feet;

therefore it has $\frac{22c}{15b}$ velos.

7. The **measure** of the velocity of a point (when a velo is the unit) is the *number* of velos which it has.

This is the **ratio** of the number of feet passed over in any interval to the number of seconds in that interval.

For, with the notation of Art. 6, $s = vt$;

therefore $$\mathbf{v} = \frac{\mathbf{s}}{\mathbf{t}}.$$

Hence, *the uniform velocity of a point* is *measured* by the **rate** at which it passes over distance per unit interval of time; in other words, by the **number** of units of distance which it passes over in unit interval.

This is what is meant when it is said that *The uniform velocity of a point is numerically equal to the distance it passes over divided by the interval occupied.*

EXAMPLES. II.

N. B. *The velocities given are uniform.*

1. Express in velos a velocity of c miles per hour.

2. A point has v velos; how far does it go in h hours?

3. A point has k velos; how long does it take to go m yards?

4. A point goes m yards in t seconds; how many miles does it go in h hours supposing its velocity uniform?

5. A point goes λ inches in 1 minute; how far does it go in k days at the same rate?

6. A point goes k feet in t seconds; how long does it take to go m miles with the same velocity?

7. A train goes n miles in k hours; how far does it go in t seconds?

AVERAGE VELOCITY.

8. DEF. The **average velocity** of a moving point *for a stated interval* (during which interval it is *not* moving *uniformly*), is the velocity of another point which, *moving uniformly*, passes over an equal distance in the same interval.

Example. A point moves over 6 ft., 7 ft., 8 ft. and 9 ft. *respectively, in four consecutive* seconds; *find its* average velocity *for the four seconds.*

In 4 seconds it passes over $(6+7+8+9)$ ft.; that is, 30 ft.

Thus, in 4 seconds a point having the required average velocity passes over 30 feet;

therefore in 1 second this point passes over $7\frac{1}{2}$ feet;

that is, the required average velocity is $\underline{7\frac{1}{2}\text{ velos}}$.

EXAMPLES. III.

1. A point passes over 9 ft., 10 ft., 11 ft. and 12 ft. in 4 consecutive seconds; find its average velocity for the 4 seconds.

2. Find the average velocity of the point in Question 1 (i) for the first 3 seconds, (ii) for the last 3 seconds.

3. A point passes over 20 yds., 24 yds., 28 yds., 32 yds. and 36 yds. in 5 consecutive seconds, shew that the average velocities for the 5 seconds, for the 3 middle seconds and for the 1 middle second are all equal.

4. A point has an average velocity of 2 velos, 3 velos, 4 velos, 5 velos, and 6 velos respectively in 5 consecutive intervals of 5 seconds each; find the average velocity for the 25 seconds; how far does the point go in the 25 secs.?

5. A point moves over a ft., $2a$ ft., $3a$ ft. and $4a$ ft. respectively in 4 consecutive intervals of b seconds each; find its average velocity for the whole time.

6. A point has an average velocity of $13v$ velos, $11v$ velos, $9v$ velos, $7v$ velos and $5v$ velos respectively in 5 consecutive intervals of t seconds each; find the average velocity for the $5t$ seconds.

7. A point has an average velocity of 3 velos for 2 secs., 6 velos for the next 4 secs., 10 velos for the next 4 secs. and 15 velos for the next 6 secs.; shew that in the 16 seconds it passes over the same distance as a point which has an average velocity of 3, 5, 7, 9, 11, 13, 15, 17 velos respectively in 8 consecutive intervals of 2 seconds each.

VELOCITY WHICH IS NOT UNIFORM.

9. Consider a point in motion *not* moving *uniformly*.

Its average velocity during any interval varies directly as the distance passed over in that interval and inversely as the length of the interval.

Its average velocity is different for different intervals.

It is the *average* velocity of a train for a certain interval, which is found by noting the number of seconds occupied by the train in passing successive quarter-mile posts; but it is not the average velocity which is meant when we speak of the velocity of a point **at** a certain instant.

10. The student must carefully distinguish between the velocity of a point **at** *a given* **instant** and its average velocity **for** *a given interval.*

Consider two very long trains moving on parallel lines in the same direction. Let one of the trains (P) be moving with a constant velocity,

say, 20 velos; let the other train (Q) start from rest and move gradually faster and faster. A passenger in train P will see his train gain on train Q at first; but as time goes on, the train Q goes faster, until at last there comes an instant at which the two trains are moving at exactly the same rate and neither is gaining on the other; this is only for an instant; it occurs *at* a certain instant. The train Q immediately begins to go faster than the train P; but there is a particular instant (the instant already referred to) *at* which neither train is going faster than the other and the two trains are *relatively at rest*. At that instant we say, that the velocity of the train Q is equal to the velocity of the train P; that is, the velocity of the train Q *at* that instant is 20 velos; hence,

DEF. The velocity of a moving point (which is not moving uniformly) **at a given instant**, is the velocity of another point which, moving with uniform velocity in the same direction, is at that instant at rest relatively to the first moving point.

UNIFORMLY INCREASING VELOCITY.

11. The simplest case of a point moving with velocity which is not uniform, is that of a point whose velocity *steadily* increases or whose velocity *steadily* decreases.

When a point is said to have velocity which is increasing or decreasing, the point does not move for an interval with uniform velocity, and then suddenly move with a different velocity; but the velocity *at* any instant gradually increases or decreases as time goes on.

12. DEF. A point moves with **uniformly increasing** velocity when its velocity is increased by equal increments in equal intervals of time, no matter how large or how small these equal intervals may be.

Uniformly increasing velocity is that which *grows* steadily; it does not increase by jerks or steps.

Suppose that a point P is moving with uniformly increasing velocity; suppose that at a certain instant its velocity is 100 velos, and that after an interval of 1 second its velocity is 101 velos; then at the end of one more second its velocity will be 102 velos; at the end of another half-second its velocity will be $102\frac{1}{2}$; and so on. Its velocity increases *at the rate* of 1 velo per second.

Example. A point moving with uniformly increasing velocity has velocities 20 velos *and* 28 velos *respectively at the beginning and at the end of an interval of* 12 seconds; *find its velocity* 3 seconds *before the beginning of the interval and* 3 seconds *after the end of the interval.*

In 12 seconds the velocity increases by 8 velos; therefore
in 1 second the velocity increases by $\frac{8}{12}$, or $\frac{2}{3}$ velo.

Hence, 3 seconds before the instant at which it has 20 velos it has $(20 - 3 \times \frac{2}{3})$ velos, or 18 velos. And 3 seconds after the instant at which it has 28 velos, it has $(28 + 3 \times \frac{2}{3})$ velos, or 30 velos.

13. A similar definition applies to the expression **uniformly decreasing** velocity.

Example. A point moving with uniformly decreasing *velocity has at a certain instant* 30 velos; *after* 5 seconds *it has* 20 velos; *when will it come to rest? when will it again have* 30 velos?

The velocity diminishes by 10 velos in 5 seconds,
therefore it diminishes by 30 velos in 15 seconds.

When it has diminished by 30 velos it has 0 velo and is at rest. Therefore it comes to rest in 15 secs. from the instant at which it has 30 velos.

[When the velocity of a point is uniformly *decreasing*, we may consider the motion to be such that velocity in the **opposite direction** to its initial velocity is being continuously added. This continuous growth of *negative* velocity does not necessarily cease when the point comes to rest, Hence,] the velocity is still continuously *diminished* at the rate of 2 velos per second after the point has come to rest; therefore after 15 seconds more it will be moving in the *opposite direction* with 30 velos.

EXAMPLES. IV.

N. B. *In the following examples the motion is in each case that of uniformly increasing or decreasing velocity.*

1. A point has 8 velos at the beginning and 9 velos at the end of a certain second; how many velos has it after 4 more seconds?

2. In a certain interval of half a minute the velocity of a point increases from 10 velos to 100 velos; what was its velocity at the middle of the interval?

3. In 1 hour the velocity of a point decreases from 300 velos to 120 velos; what was its velocity at the end of each quarter of that hour?

4. At noon a point is moving with 20 velos; at 4 P.M. it has 100 velos; what velocity has it at 2.15 P.M.?

5. At 2 o'clock a point has 7 velos; at 2.45 it has 142 velos; what has it at 3.30?

6. A train is being pulled up, and moves with uniformly decreasing velocity; at a certain instant it is going 60 miles per hour, after 18 seconds it has 80 velos; when will it stop?

7. A point which started from rest has after 4 seconds a velocity of 100 yds. per minute; when will it have 10 velos? and when will it be going at the rate of 60 miles an hour?

8. A point at a certain instant has 11 velos; 2 minutes later it is moving with a velocity of 30 miles an hour; when was it at rest?

9. At a certain instant a point has 128 velos and its velocity is decreasing at the rate of 32 velos per second; when will it come to rest? and how long will it be before it again has 128 velos?

10. A point moving at a certain instant at the rate of 11 ft. per second has its velocity increased by (7 ft. per second) per second; when will it be moving at the rate of 60 miles an hour?

11. A point which is moving at a certain instant with a velocity which is decreasing at the rate of (32 ft. per second) per second comes to rest after an interval of 3 seconds; what was its velocity at the beginning of the 3 seconds? and what at the middle of the 3 seconds? and what was it after 3 seconds more?

12. A point whose velocity is increasing at the rate of 32 ft. per second per second is moving at a certain instant vertically downwards with a velocity of 112 ft. per second; when was it at rest? and when was it moving vertically upwards with a velocity of 112 ft. per second?

14. When a point is moving with velocity which is uniformly increasing, its *average* velocity for a given interval of time is greater than the velocity which the point has at the *beginning* of the interval and is less than the velocity which the point has at the *end* of the interval.

For its velocity at the beginning of the interval is less than its velocity at any other instant in the interval; and its velocity at the end of the interval is greater than its velocity at any other instant in the interval.

15. The same thing may also be stated thus;

Let a point, moving with uniformly increasing velocity, at the beginning of a certain interval of t seconds have u velos, and let the point at the end of the interval have v velos; then, the distance passed over by the point in this particular interval of t seconds is greater than $t \times u$ feet and is less than $t \times v$ feet.

Example i. *At a certain instant a point has* 8 velos *and its velocity is uniformly increasing* at the rate of 2 velos per second; *shew that in the next* four seconds *it passes over more than* 44 feet *and less than* 52 feet.

Its velocity at the *beginning* of each second is 8, 10, 12, 14 velos respectively; therefore it passes over *more* than $(8+10+12+14)$ feet in these 4 seconds.

Its velocity at the *end* of each second is 10, 12, 14, 16 velos respectively; therefore it passes over *less* than $(10+12+14+16)$ feet in these 4 seconds.

Example ii. *Shew by taking the velocity of the point in* Example i *at the beginning and at the end of each* quarter second, *that the distance passed over in the* 4 seconds *lies between* 47 feet *and* 49 feet.

At the *beginning* of each $\frac{1}{4}$ second the velocities are

$$8,\ 8\tfrac{1}{2},\ 9,\ 9\tfrac{1}{2},\ 10,\ 10\tfrac{1}{2} \text{ etc.}\ldots\text{up to } 15\tfrac{1}{2}, \text{ velos, respectively,}$$

and the distance passed over is therefore *greater* than

$$\tfrac{1}{4}\{8+8\tfrac{1}{2}+9+9\tfrac{1}{2}+\ldots\ldots\ldots\ldots\ldots+15\tfrac{1}{2}\} \text{ feet,}$$

that is, *greater* than (the sum of the series in A.P., viz.) 47 feet.

At the *end* of each $\frac{1}{4}$ second the velocities are

$$8\tfrac{1}{2},\ 9,\ 9\tfrac{1}{2} \text{ etc.}\ldots\text{up to } 16, \text{ velos, respectively,}$$

and the distance passed over is therefore *less* than

$$\tfrac{1}{4}\{8\tfrac{1}{2}+9+9\tfrac{1}{2}+\ldots\ldots\ldots\ldots\ldots+16\} \text{ feet,}$$

that is, *less* than 49 feet. Q. E. D.

EXAMPLES. V.

N. B. *The following velocities are each uniformly increasing or decreasing.*

1. At a certain instant a point has 8 velos and its velocity is increasing uniformly at the rate of 1 velo per second; shew by considering the velocity at the beginning and at the end

(i) of each second, that the distance passed over in the next 10 seconds is greater than 125 ft. and less than 135 ft.

(ii) of each $\frac{1}{10}$th of a second, that the distance passed over in the 10 seconds is greater than $129\frac{1}{2}$ ft. and less than $130\frac{1}{2}$ ft.

2. At a certain instant a point has 120 velos and its velocity is decreasing at the rate of 32 velos per second; shew by considering the velocity at the end of each $\frac{1}{32}$th of a second that the distance passed over in the next second is less than $104\frac{1}{2}$ ft. and greater than $103\frac{1}{2}$ ft.

3. At a certain instant a point has 16 velos and its velocity is increasing at the rate of 32 velos per second; shew by dividing the interval of time into $\frac{1}{32}$ths of a second that the distance passed over in the next 3 seconds cannot differ from 192 feet by more than 18 inches.

4. At a certain instant a point is at rest and velocity is given to it uniformly so that at the end of 1 second it has 32 velos; shew as in example **3**, that in the 3rd second of its motion the distance which it passes over does not differ from 80 feet by more than 6 inches.

5. At a certain instant a point has u velos, and its velocity is increasing at the rate of 32 velos per second; shew, by dividing the next 8 seconds into $8n$ equal parts, that the distance passed over in those 8 seconds cannot differ from $(8u+1024)$ feet by more than $\frac{128}{n}$ feet.

6. At a certain instant a point has u velos, and its velocity is increasing at the rate of α velos per second; shew, by dividing the next t seconds into n equal parts, and considering the velocity at the beginning and the end of these n intervals, that the distance passed over in t seconds cannot differ from $(u+\frac{1}{2}\alpha t)\times t$ feet by more than $\frac{\alpha t^2}{2n}$ feet.

16. It is of the greatest importance that the student should mentally realize the motion of a point moving with uniformly increasing (or decreasing) velocity.

Consider again the case of two very long trains A and B moving on parallel lines in the same direction; suppose them to be passing a certain station H,—the train A with a constant velocity 20 velos,—the train B with a constantly increasing velocity which, when the engine is at H, is 4 velos.

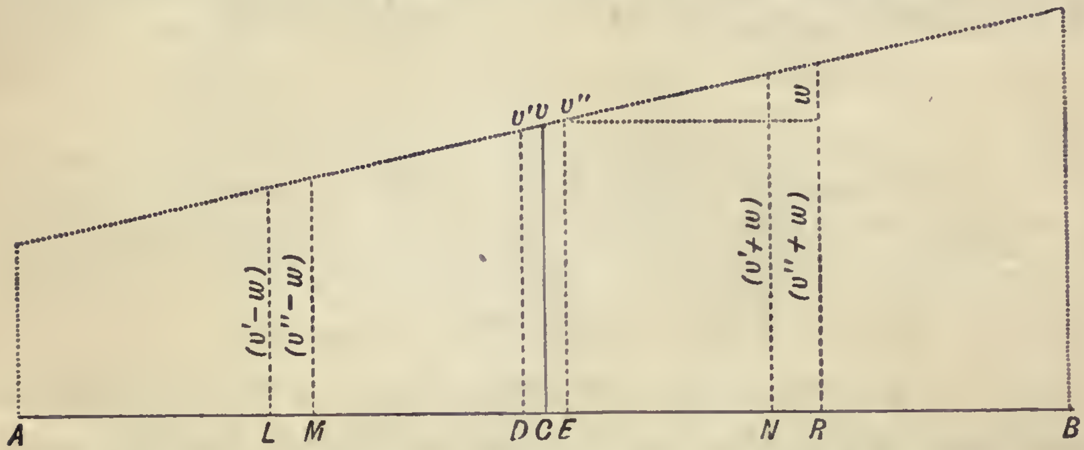
At H suppose the engines are abreast; at first A will gain upon B; but it will gradually gain more and more slowly, until an instant arrives when the two trains are relatively at rest; after which time B will at once begin to gain upon A; and will gradually gain at a faster and faster rate.

Hence an instant arrives when the engines are again abreast; suppose this to occur at a place K; then each train has passed over the distance from H to K in the same interval; hence the average velocity of B *for the time occupied in going from H to K* is the velocity of A, viz. 20 velos.

We proceed to prove that *the instant at which* these two trains are relatively at rest, *is at the middle of the interval of* time which they each take in going from H to K. (So that the engine of the train A, which moves with uniform velocity, is *midway* between the stations, while the engine of the train B has not yet reached the middle point.)

The proof given on the next page may be deferred if it is thought desirable. An alternative proof is given in Art. 24.

17. PROP. *When the velocity of a point is uniformly increasing, the* **average velocity** *of the point for any given interval of time is equal to the velocity which the moving point has* **at the middle of that interval of time.**



Let the given interval of time t seconds be represented by the length of the finite straight line AB.

Divide the t seconds into any *odd* number n of smaller equal intervals, each containing τ seconds.

Let DE represent the *middle* interval of τ seconds; and let C be the middle instant of AB, and therefore of DE. Take a pair of the little intervals LM and NR, one on each side of C and equidistant from C.

Let the moving point *at* the instant D have v' velos; *at* the instant C, v velos; and at the instant E, v'' velos; let w be the number of velos added on in the interval LD.

At the instant L the moving point has $(v' - w)$ velos; at the instant N it has $(v' + w)$ velos; at the instant M it has $(v'' - w)$ velos; at the instant R it has $(v'' + w)$ velos. [Since the number of velos which it has increases uniformly with the time, and $LD = ME = DN = ER$.]

The distance passed over by the point in the intervals LM and NR together, is therefore greater than $\{(v' - w)\tau + (v' + w)\tau\}$ feet, and less than $\{(v'' - w)\tau + (v'' + w)\tau\}$ feet; that is, greater than $2\tau v'$ feet, and less than $2\tau v''$ feet.

The same is true for any other pair of little intervals in

AD and *EB*. Hence since there are $\frac{1}{2}(n-1)$ intervals on either side of *DE*, the distance passed over in *AD* and *EB* together is greater than $(n-1)\tau v'$ and less than $(n-1)\tau v''$.

Also, in the middle interval *DE*, the distance passed over is greater than $\tau v'$ ft. and less than $\tau v''$ ft.

Hence, the whole distance passed over in all the n intervals is greater than $n\tau \times v'$ ft. and less than $n\tau \times v''$ ft.; that is, greater than tv' feet and less than tv'' feet.

This is true however great the odd number n may be.

Let n be increased without limit; then τ is diminished without limit and v' and v'' each approach v as their limit. Therefore the distance passed over in the whole interval t seconds (which distance is *between* tv' feet and tv'' feet) *cannot be other than* tv feet.

In other words, its average velocity for the time t seconds is v velos; and this is the velocity which the point has at C, the middle of that interval of t seconds. Q. E. D.

Example i. *At a certain instant a point has* 18 velos, *and it is moving with uniformly increasing velocity, so that* 1 second *later it has* 20 velos; *how far does it go in that second?*

The average velocity for the 1 sec. is the same as the velocity which the point has at the middle of the interval, that is, after half a second.

Since in 1 second the velocity is increased by 2 velos,
therefore in $\frac{1}{2}$ second the velocity is increased by 1 velo;
hence the average velocity for the 1 second is $(18+1)$, or, 19 velos.

In 1 second it passes over 19×1 ft., or 19 feet.

Example ii. *How far does the point in* Example i. *go in* 8 seconds?

The middle of the interval of 8 seconds is 4 seconds after the time when it has 18 velos. In 1 second the velocity is increased by 2 velos,
therefore in 4 seconds the velocity is increased by 8 velos;
hence, the average velocity for the 8 seconds is $(18+8)$ or 26 velos;
therefore, the distance passed over is 26×8 ft.; that is, 208 feet.

Example iii. *A point, moving with uniformly decreasing* velocity, *passes over* 294 ft. *in a certain interval of* 7 secs.; *at the beginning of the* 7 seconds *it has* 49 velos; *find its* velocity *after* 3 seconds *more.*

Since it passes over 294 ft. in the 7 seconds, its average velocity

[Art. 8] is 42 velos; its actual velocity at the middle of the 7 secs. is equal to this average velocity; hence, since at the beginning of the 7 seconds it has 49 velos, and $3\frac{1}{2}$ seconds later it has 42 velos, therefore its velocity decreases by 7 velos in $3\frac{1}{2}$ secs.; or, by 2 velos per second;

hence, after 10 seconds it will have $(49 - 20)$ or 29 velos.

EXAMPLES. VI.

N. B. *In the following examples the velocity is uniformly increasing (or decreasing).*

1. A point has 100 velos at 12 o'clock, and its velocity increases to 104 velos in 1 second; find

(i) how far it goes in 1 second, (ii) how far it goes in 2 seconds,
(iii) how far it goes in the first 30 secs. after 1 min. past 12 o'clock,
(iv) how far it goes between 12.1′ and 12.2′ o'clock.

2. At a certain instant a point has 16 velos, and its velocity increases at the rate of 32 velos per second; shew that in the next three seconds it passes over 192 ft. [See Ex. V. 3.]

3. At a certain instant a point has 120 velos, and its velocity is decreasing at the rate of 32 velos per second; shew that in the next second it passes over 104 ft. [See Ex. V. 2.]

4. A point in a certain interval of 3 seconds passes over 300 feet; in the next 4 seconds it passes over 764 ft.; shew that its velocity is increasing at the rate of 26 velos per second.

5. A point in a certain interval of 5 seconds passes over 125 feet; in the next 8 secs. it passes over 304 ft.; when did it start from rest?

6. A point is observed to pass over 120 ft., 129 ft., 138 ft. respectively in 3 consecutive seconds; is this consistent with uniformly increasing velocity?

7. Starting from rest, in the 5th second of its motion, a point passes over 144 ft.; find its velocity after 10 seconds from rest.

8. A point passes over 144 ft. in a certain interval of 2 secs. and comes to rest after 3 secs. more; how much further did it move?

9. A point starts from rest, and after 10 seconds it has 25 velos; how far does it go in the 20th second of its motion?

10. A point passes over 24 ft. in a certain second; at the end of that second it has 28 velos; how far does it go in the next second? and when is it at rest?

11. A point passes over 144 ft. in two seconds and over 96 ft. in the next two seconds; when will it come to rest? and how far does it move before doing so?

12. A train goes 10 miles in a quarter of an hour, having started from rest; shew that if its velocity has been uniformly increasing, it is moving at the end of that quarter of an hour with a speed of 80 miles per hour.

UNIFORM ACCELERATION.

18. DEF. When a point is moving with uniformly increasing velocity, its **acceleration** is *that* which varies directly as the *increase* of velocity in a given interval, and inversely as the interval required for a given increase of velocity. [Compare Arts. 3 and 4.]

Thus, the acceleration of a point is doubled, or trebled etc. when the velocity added in a given interval is doubled, or trebled etc.; and it is halved, or divided by three, etc. when the interval (in which a given velocity is added) is doubled, or trebled etc.

19. We choose as our **unit** *acceleration*, the acceleration of a point which, moving with uniformly increasing velocity, has its velocity increased by 1 velo in the course of each second.

We shall call this *unit acceleration* a **celo**.

Thus, *a celo* is (*a foot per second*) *per second*.

In the present work we consider only **uniform** acceleration. So that when a point is said to have a certain number of celos, it is implied that the point has that uniform acceleration *throughout* the interval under consideration.

NOTE. It is most important to notice that a point which has acceleration requires an interval of time in which to increase its velocity by any definite amount.

Example i. A point which has 6 velos added to its velocity in the course of each second, has six times the acceleration of a point having the unit acceleration; hence it has 6 *celos*. The same point has 1 velo added in the course of the $\frac{1}{6}$th part of a second.

Example ii. *A point which has* 6 celos *has at a certain instant* 12 velos; *how many* velos *will it have* 10 seconds *later?*

In the course of 1 second 6 velos are added,
therefore in 10 seconds 60 velos are added.

So that after 10 secs. the point has (12 + 60) velos, or 72 velos.

Example iii. *How far does the point in* Ex. ii. *go in those* 10 secs.?

The average velocity of the point for the 10 seconds is that which it has at the middle of the interval; that is, $(12+5\times 6)$ velos, or 42 velos; therefore, in 10 seconds the point goes 42×10 ft., or 420 feet.

Example iv. *A point which has* 32 celos *starts* from rest; *how far will it go in the* 4th second *of its motion? and how far in the* 12th second?

At the middle of the 4th second, that is, after $3\frac{1}{2}$ secs., it has

$3\frac{1}{2}\times 32$ velos, or 112 velos; hence,

for the 4th second its average velocity is 112 velos;
therefore, in that second it passes over 112×1 ft., or 112 ft.

Again, at the middle of the 12th second; that is, at the end of $11\frac{1}{2}$ secs. it has $11\frac{1}{2}\times 32$ velos, or 368 velos; therefore,
for the 12th second its average velocity is 368 velos;
so that in that second it passes over 368 ft.

Example v. *A point which at a certain instant has* u velos, *has* u' velos *added on in the next* t seconds; *what is its* acceleration?

When 1 velo is added in 1 second the point has 1 celo;
therefore, when 1 velo is added in t seconds the point has $\frac{1}{t}$ celos,
and when u' velos are added in t seconds the point has $\frac{u'}{t}$ celos.
Therefore the required acceleration is $\frac{u'}{t}$ celos. Hence

20. The *measure* of the acceleration of a point is the **ratio** of the *number* of velos added in an interval to the *number* of seconds in that interval.

In other words, the acceleration of a point is *measured* by the **rate** at which its velocity increases per unit-interval of time.

It is numerically equal to the **number** of units of velocity added on per unit-interval.

21. When the velocity of a point is *decreasing* uniformly it is said to have **negative acceleration.**

When the acceleration of a point is opposite in direction to its velocity it is often called **retardation.**

Suppose a point which has a uniform acceleration to have at a certain instant 32 velos and at the end of 1 second to have 30 velos, we say that it has -2 celos. This is consistent with the actual result and is a very convenient way of speaking. [See Art. 13.]

This point at the end of 5 seconds will have $(32 - 2 \times 5)$ velos.

EXAMPLES. VII.

N. B. *In these examples the motion in each case is uniformly accelerated in one straight line; so that when a point is said to have a certain number of celos, it retains that acceleration* throughout *the motion under consideration.*

1. A point which has 32 celos, has at a certain instant 16 velos, find (i) its velocity after 2 seconds, (ii) its velocity after 5 seconds,
(iii) how far it goes in the first 4 seconds,
(iv) how far it goes in the first 20 seconds,
(v) when it will have 1100 velos, (vi) when it was at rest.

2. At a certain instant a point has 200 velos; 5 seconds later it has 300 velos; find its acceleration.

3. A point which has 32 celos passes over 144 ft. in a certain interval of 3 seconds; shew that it started from rest.

4. A point which has 32 celos passes over 144 ft. in a certain interval of 3 seconds; how far will it go in the next 4 seconds?

5. In a certain interval of 5 seconds a point passes over 250 ft. and in the next 8 seconds it passes over 608 ft.; find its acceleration.

6. A point which has a certain acceleration is at a certain instant at rest; during the tenth second from rest it passes over 304 ft.; find
(i) its acceleration,
(ii) the distance it passes over in the 5th second reckoning from the instant at which it was at rest.

7. A point which has -5 celos, has at a certain instant 30 velos; find (i) its velocity at the end of 3 seconds,
(ii) when and where it comes to rest,
(iii) the distance passed over in the first 5 seconds,
(iv) the ratio of the distance passed over in the first 3 seconds to that passed over in the next 3 seconds.

8. In a certain interval of 3 secs. a point passes over 288 ft.; in next 2 secs. it passes over 32 ft.; find
(i) its acceleration, (ii) when it is at rest,
(iii) where it is after 3 more seconds,
(iv) when it has 144 velos, (v) when it has -144 velos.

9. A point passes over h feet in t seconds; in the next $2t$ seconds it passes over k feet; find its acceleration.

10. A point passes over h feet in 2 seconds; after an interval of t seconds it is observed to pass over k feet in 2 seconds; find its acceleration.

22. PROP. *Let a point have* u velos *at the beginning of a certain interval of* t seconds; *let it have* α celos *throughout the interval; and let* s feet *be the distance it passes over in the* t seconds; *then* $\mathbf{s = ut + \frac{1}{2}\alpha t^2}$.

For, the *average velocity* of the point for the t seconds is that which it has *at* the *middle* of the interval; that is, the velocity which it has after $\frac{1}{2}t$ seconds.

But α velos are added to its velocity per second; therefore, ($\frac{1}{2}t$ times α) velos are added on in $\frac{1}{2}t$ seconds; so that its average velocity for the t seconds is $(u + \frac{1}{2}\alpha t)$ velos.

Therefore, the distance passed over in the t seconds is

$$[(u + \tfrac{1}{2}\alpha t) \times t] \text{ feet};$$

hence, $$s = ut + \tfrac{1}{2}\alpha t^2. \qquad \text{Q. E. D.}$$

N. B. In future, when using the letters s, u, α or t we shall always suppose them to have the meaning here given.

23. The results of Arts. 17 and 22 may also be proved by the method of Examples V. **6** as follows [see Ex. IX. **23**].

PROP. *To prove, with the usual notation, that*

$$s = ut + \tfrac{1}{2}\alpha t^2.$$

Let the interval t seconds be divided into n equal intervals each of τ seconds; so that, $n\tau = t$.

The velocities at the beginning of each of these n intervals are respectively

$$u \text{ velos},$$
$$(u + \alpha\tau) \text{ velos},$$
$$(u + 2\alpha\tau) \text{ velos},$$
$$\ldots$$
$$\ldots$$
$$(u + \overline{n-1}\,\alpha\tau) \text{ velos}.$$

The velocities at the end of each of these n intervals are respectively

$$(u + \alpha\tau) \text{ velos},$$
$$(u + 2\alpha\tau) \text{ velos},$$
$$(u + 3\alpha\tau) \text{ velos},$$
$$\ldots$$
$$\ldots$$
$$(u + n\alpha\tau) \text{ velos}.$$

Let s' and s'' respectively be the distances which a moving point would pass over in these n intervals supposing its velocity during each interval to be constant and equal in the case of s' to that of the above moving point at the beginning of the interval and in the case of s'' to that at the end of the interval.

Then by Art. 14, s lies between s' and s''.

But $s' = \{u\tau + (u + a\tau)\,\tau + (u + 2a\tau)\,\tau + \dots (u + \overline{n-1}\,a\tau)\,\tau\}$

$$= \tfrac{1}{2}n\,\{2u\tau + \overline{n-1}\,a\tau^2\} = un\tau + \tfrac{1}{2}an^2\tau^2 - \tfrac{1}{2}\frac{an^2\tau^2}{n}$$

$$= ut + \tfrac{1}{2}at^2 - \tfrac{1}{2}\frac{at^2}{n} \qquad [\text{since } n\tau = t].$$

And $s'' = \{(u + a\tau)\,\tau + (u + 2a\tau)\,\tau + \dots + (u + na\tau)\,\tau\}$

$$= \tfrac{1}{2}n\,\{2(u + a\tau)\,\tau + (n-1)\,a\tau^2\}$$

$$= ut + \tfrac{1}{2}at^2 + \tfrac{1}{2}\frac{at^2}{n}.$$

This is true however small τ, that is, however great n may be; and $\tfrac{1}{2}\frac{at^2}{n}$ diminishes as n increases.

Let n be increased without limit; and then s' and s'' both approach the limit $ut + \tfrac{1}{2}at^2$.

Therefore the true value of s is $ut + \tfrac{1}{2}at^2$. Q. E. D.

*24. The diagram of Art. 17, which may be called a **Diagram of Velocity**, will be found to be useful in solving questions on motion.

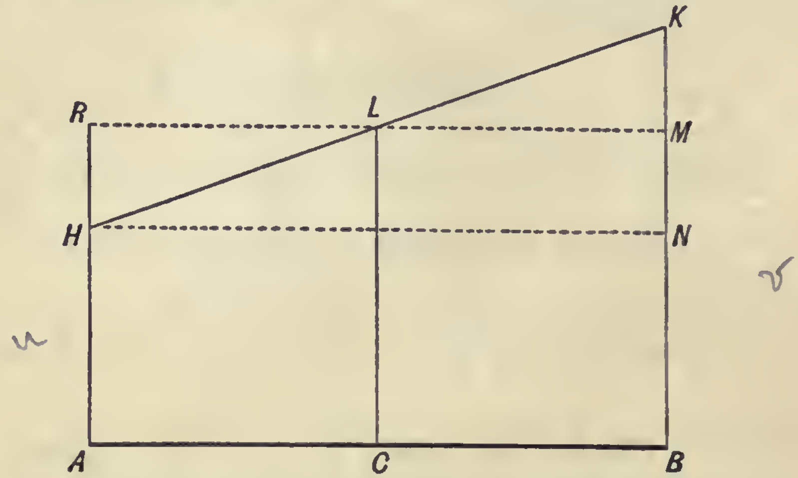
We draw a horizontal line so that its length records the interval; this horizontal line may be looked upon as a part of the rim of a clock, straightened.

Different *points* on this line indicate different *instants*.

We record the *number* of velos which a point has at a particular instant by the *length* of a line drawn, from the point indicating that instant, at right angles to the horizontal line.

It must be most carefully noticed that there is no *line* in the diagram which indicates the *distance* passed over.

Example i. *Prove that the* velocity, at the middle of an interval, *of a point moving with uniformly increasing velocity, is* half the sum of the velocity at the beginning and the velocity at the end of the interval.



Let AB represent the interval t seconds; let AH drawn perpendicular to AB represent u velos; let BK represent v velos; bisect AB in C; join HK; draw CL parallel to AH.

Then, since the velocity of the point is *uniformly* increasing, the line drawn from any point in AB, parallel to AH, intercepted between AB and HK represents the velocity at the instant indicated by the point in AB.

Therefore CL represents the velocity at C, that is, the velocity at the middle of the interval AB.

Now draw RLM and HN parallel to AB.

Then since $RL = LM$, $\therefore\ HR = MK$,

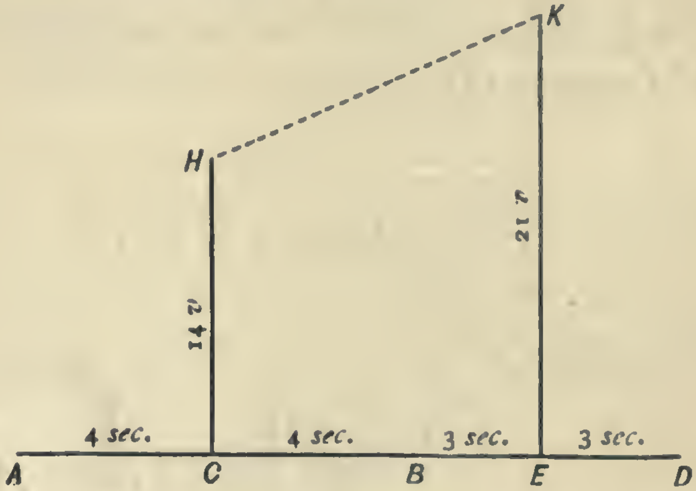
and $2CL = AR + BM = AH + MK + BM = AH + BK$;

$\therefore$ the velocity at $C = \frac{1}{2}(u+v)$ velos. Q.E.D.

Hence, by Art. 17 the average velocity for the t seconds is $\frac{1}{2}(u+v)$. Whence it follows that $s = \frac{1}{2}(u+v)t$.

If each linear unit in AB represents a second and each linear unit in AH represents a velo, then the *acceleration* of the moving point is numerically equal to the ratio of KN to HN in the above diagram. The *distance* passed over is numerically equal to the product of CL and AB; that is, to the *area* of $ARMB$, that is to the *area* of $AHKB$.

Example. A point passes over 112 feet *in* 8 seconds *and it passes over the next* 126 feet *in* 6 seconds; *find its* acceleration; *find also* how long *it will take to go the next* 162 feet.



Draw the velocity diagram; let AB represent 8 secs.; let BD represent 6 seconds.

The velocity at C, the middle instant of AB, is $\frac{112}{8}$ velos $=14$ velos (Art. 17); represent this by CH.

The velocity at E, the middle instant of BD, is $\frac{126}{6}$ velos $=21$ velos; represent this by EK. Join HK.

The velocity is uniformly increasing. It increases by $(EK-CH)$, or by 7 velos in CE, that is, in $(4+3)$ seconds, or, in 7 secs.

Therefore the point has 1 celo.

At the instant D the velocity is $(21+3)$ or 24 velos.

Hence, if the point goes the next 162 ft. in x seconds,

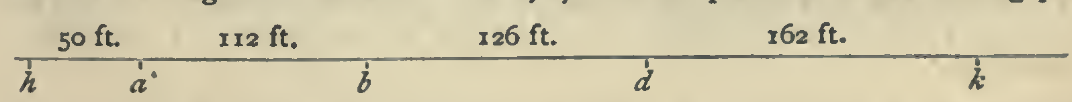
$$162=24x+\tfrac{1}{2}x^2; \quad \text{or,} \quad x^2+48x=324.$$

Whence $\qquad x=6$ or -54†...

That is the point takes 6 seconds to go the next 252 ft.

† This result indicates that at an instant 54 seconds before the instant D (from which instant the interval x seconds is measured) the distance of the moving point (P) was 162 feet from its position at the instant D. Here we assume that the moving point P has 1 celo for an unlimited length of time.

Draw the diagram of distance. Let a, b, d be the positions of the moving point



at the instants A, B, D. Then $ab=112$ ft., $bd=126$ ft.; let $dk=162$ ft.; let $ha=50$ ft. Then it will be found that 54 seconds before D the moving point was at k, moving in the direction kd with 30 velos; 30 seconds later it came to rest at h; 10 seconds later it was at a; 14 seconds later it was at d; 6 seconds later it was again at k. During all this time it has 1 celo in the direction hk.

25. *To prove that, if v be the* number of velos *which the moving point has at the end of the interval* t seconds, *then* $$2\mathbf{as} = \mathbf{v}^2 - \mathbf{u}^2.$$

Since a is the acceleration, the additional velocity at the end of t seconds is at velos; $\therefore\ v = u + at$,

hence $$a = \frac{v - u}{t},$$

also $$s = ut + \tfrac{1}{2}at^2 = \tfrac{1}{2}(v + u)\,t;$$

$$\therefore\ 2as = (v - u)(v + u) = v^2 - u^2.$$

26. These three results are of great importance; viz.

$$\mathbf{v} = \mathbf{u} + \mathbf{at} \qquad \text{(i)},$$

$$\mathbf{s} = \mathbf{ut} + \tfrac{1}{2}\mathbf{at}^2 = \tfrac{1}{2}(\mathbf{u} + \mathbf{v})\,\mathbf{t} \qquad \text{(ii)},$$

$$\mathbf{as} = \tfrac{1}{2}\mathbf{v}^2 - \tfrac{1}{2}\mathbf{u}^2 \qquad \text{(iii)},$$

where **t** is the *number* of seconds in the interval under consideration,

u is the *number* of velos which the moving point has at the beginning of the t seconds,

v is the *number* of velos which the point has at the end of the t seconds,

$\boldsymbol{a}$ is the *number* of celos which the point has throughout the t seconds,

s is the *number* of feet between the positions† of the point at the beginning and at the end of the t seconds.

† NOTE. The number of feet given by the formula $s = ut + \frac{1}{2}at^2$ when a and u are of opposite sign is nct always the distance *passed over* by the point, in the sense of *the number of feet travelled*. It is the *resultant* distance of the point from the initial point. In fact s gives the *position* of the point at the *end* of the interval t seconds.

Thus in the above Example i. $s = 1600$ when $t = 5$ and also when $t = 20$, but during the interval 20 seconds the point travels many more than 1600 feet; it travels 1600 feet in 5 seconds, then goes 900 feet further in $7\frac{1}{2}$ secs. and in the next $7\frac{1}{2}$ secs. comes back 900 feet; so that at the end of 20 secs. it is again at a distance 1600 feet from the initial point.

Example i. How long *will a point which has* -32 celos *and which when at a certain point has* 400 velos, *take to go a distance of* 1600 feet *from that point?*

Let the required interval be x seconds; then since $s=ut+\frac{1}{2}at^2$, therefore, $1600=400x-16x^2$, or, $x^2-25x+100=0$, or, $(x-20)(x-5)=0$. So that, either $x=20$, or $x=5$.

Thus the required interval is either 5 *seconds*, or 20 *seconds.*

The point has -32 celos; that is, its velocity is decreasing. In 5 seconds it passes over 1600 ft.; then it goes on, comes to rest, and returns to the same point, after 15 seconds more.

Example ii. *A point which has* 5 celos *has initially* 20 velos; how far *will it go in attaining a velocity of* 100 velos?

Let the required distance be x feet; then since

$$as=\tfrac{1}{2}v^2-\tfrac{1}{2}u^2,$$

therefore, $$5x=\tfrac{1}{2}(100^2-20^2),$$

or, $$5x=5000-200.$$

that is, $$x=960.$$

The required distance is 960 *feet.*

Example iii. *A point which has an* acceleration *of* 1 mile per minute per minute *is moving* at a certain instant *with a* velocity *of* 15 miles per hour; *how far does it go in the next minute?*

15 miles per hour = 22 velos. [Ex. ii. p. 3.]

By an acceleration of 1 mile per minute per minute, is meant that an additional velocity of one mile per minute is produced in each minute.

Now, a mile per minute $=\frac{1760}{60}\times 3$ ft. per second,

that is, 88 velos.

Hence, 88 velos are added on in the course of each minute;

that is, $\frac{88}{60}$ velos are added on per second.

Hence, 1 mile per min. per min. = $\frac{88}{60}$ celos.

Therefore the point will move over

$$\{22\times 60+\tfrac{1}{2}\cdot\tfrac{88}{60}\times(60)^2\} \text{ feet in 60 secs.;}$$

that is, $(1320+2640)$ ft., or, 3960 ft.

NOTE. When a point, which is moving with the constant acceleration a celos in a given direction, has at a certain instant the velocity v velos in the direction exactly opposite to that of the acceleration, then at the end of the interval $\frac{v}{a}$ seconds the point has velocity zero; it is at that instant at rest; it then proceeds to retrace its former path.

It is very important to notice that as the point retraces its former path its motion may be said to be the exact counterpart of its motion in tracing the path.

For, I. the velocities with which it passes and repasses any chosen point P are *equal*.

II. the instants of passing and repassing any chosen point P are *equidistant* from the instant at which it is at rest.

This follows from the fact that (i) the *average velocity* of the point for any interval of time, (ii) the velocity of the point at the middle instant of that interval, and (iii) half the sum of the velocities of the point at the beginning and at the end of that interval, are equal to each other.

Let the moving point pass the chosen point P with velocity v, and after the interval t let it repass the point P with velocity v'.

The distance passed over in the interval t is zero; [for it is $+s$ feet $-s$ feet] therefore the average velocity for the interval t is zero; therefore the velocity of the point at the middle of the interval t is zero; that is, it is [II.] *at rest* at the middle of the interval t.

Also it follows that $\frac{1}{2}(v+v')$ is zero, and $\therefore v=-v'$. [I.]

It should be noticed that the complete motion of a point which moves *continuously* with constant acceleration, and which at a certain instant is at rest at a certain point, must consist of an approach to that point along a straight line and a repassing over the same straight line with its velocity at each point equal and opposite to the velocity of its approach.

EXAMPLES. VIII.

The motion in each case is that of uniform acceleration in a straight line.

1. A point has -32 celos and at a certain instant it has 96 velos; in how many seconds will it be at a distance 144 ft. from the position which it has at that instant?

2. A point has 2 celos and initially it has 20 velos; how long will it take to go 100 ft.? and how far will it go before it attains the velocity 25 velos?

3. One point P has 32 celos and it has initially 200 velos; a second point Q has 16 celos and, initially, 1000 velos; initially they start from the same point in the same direction; when are they again together? and how far will they have gone?

4. A point which has 32 celos starts from rest; from the same place, at the same time and in the same direction another point which has 30 celos starts with an initial velocity of 40 velos; when and where will the first point overtake the second? and when will they have equal velocities?

5. A point having initially 20 velos attains a velocity of 30 miles per hour in passing over 128 ft.; find its acceleration.

6. A point after passing over a quarter of a mile with an acceleration of 20 ft. per second per second, attains a velocity of 2704 ft. per second; what was its initial velocity?

7. A train in passing over a quarter of a mile increases its velocity from 5 miles an hour to 35 miles an hour; what is its acceleration?

8. A train moving at the rate of 45 miles an hour is stopped by the continuous brake in 220 yds.; what is its 'acceleration'?

9. A train has at a certain instant a velocity of 60 miles per hour and it has a retardation of one mile per minute per minute; how far will it go before it comes to rest?

10. A point in passing over a mile has its velocity increased from a mile in 3 minutes to a mile in $2\frac{1}{2}$ minutes: how many celos has it?

11. A point passes over 100 yards in 10 seconds, and the next 100 yards in 12 seconds; when will it be 100 yards further on?

12. A point passes over 200 ft. in 12 seconds, and the next 300 ft. in 20 seconds; when will it be 400 ft. further on?

13. A stone which has -32 celos is passing a certain point; at the point 48 feet further on it has 32 velos; in how many seconds will it return to the first point?

14. A point is observed to move a distance of 48 ft. from rest in 60 seconds; what is its acceleration?

15. A point has 32 celos; it has at a certain point a certain velocity in the opposite direction to its acceleration; it returns to that point after an interval of 4 seconds; find how far it travels from that point and its velocity at that point.

16. A point having a certain constant acceleration moves in a certain second over a distance of 20 ft. and in the next second but one it moves over 128 ft.; what is its acceleration?

17. A point having a constant acceleration passes over 72 ft. while its velocity increases from 16 to 20 velos; what is its acceleration?

18. A point having a constant acceleration passes over in the fifth and the seventh second of its motion distances of 108 ft. and 140 ft. respectively; find its acceleration and its initial velocity.

19. A point having a certain constant acceleration attains a velocity of 40 miles an hour from rest in going 1 mile; the acceleration is then changed into a constant retardation so that the point comes to rest in 440 yards more; compare the magnitudes of the acceleration and retardation.

20. A point has 32 celos for 4 seconds; the acceleration is then reversed so as to be -32 celos; find how far the mass goes in 8 seconds from rest.

MISCELLANEOUS EXAMPLES. IX.

N. B. *The motion in each case is uniform acceleration in a straight line.*

1. A point moves over 7 ft., 9 ft., 11 ft., 13 ft. respectively in four consecutive seconds; find its average velocity; and supposing the velocity to be uniformly increasing, find its acceleration and its velocity at the beginning and at the end of the four seconds.

2. A point has 4 celos for 10 seconds and passes over 1000 ft. in those 10 seconds; find its initial velocity; and if it had 4 celos from rest, find when and where it started from rest.

3. At the beginning of an interval of 12 seconds a point has 40 velos, and its average velocity for the 12 seconds is 80 velos; find its acceleration.

4. A point passes over 5 ft. in one second, 16 ft. in the next 2 secs., 11 ft. in the next second, 45 ft. in the next 3 seconds; shew that this is consistent with a constant acceleration, and find the acceleration.

5. A point has -2 celos and initially it has 20 velos; find when it will be 100 ft. from the initial position.

6. A point has 32 celos and initially it has 20 velos; find how long it will take to go 1800 feet.

7. A point has -32 celos and initially it has 120 velos; find when it will be 200 ft. from the starting point.

8. A point at a certain instant has 20 velos and it goes twice as far in the third and fourth seconds together as it does in the first two seconds of its motion; find the acceleration.

9. A point having a constant acceleration starts from rest; shew that it goes in the second half of any interval (reckoning from the instant at which the point was at rest) three times as far as it does in the first half.

10. Two points move at the same instant from the same point in the same line, one with a constant velocity 4 velos, the other with 2 celos from rest; when will they be together again?

11. A point P starts from rest along a line with constant acceleration α celos, and t seconds later a point Q starts along the same line with constant velocity u velos; prove that Q will not overtake P unless u is greater than αt.

12. The line AB is 192 ft. long; P starts from A along AB with 8 velos and moves uniformly; Q starts from rest at B along BA and moves with 8 celos; find where they will meet.

13. Two points P and Q have each 32 celos in the same direction AB; P starts from A from rest and Q starts simultaneously from B having a velocity towards A just sufficient to carry it to A; find where they pass each other, if AB is 120 ft.

14. Two points P and Q have each g celos in the same direction AB; P starts from A from rest and Q starts simultaneously from B (which is a ft. from A) having a velocity of $\sqrt{(2ag)}$ velos towards A; find where they pass each other.

15. Two points P and Q have each g celos in the same direction, say vertically downwards; P starts from rest from a point A and Q starts from B, vertically below A, with a certain velocity v velos upwards. Shew that however small v may be, P will always overtake Q.

16. A point which has 32 celos moves from rest over a certain distance; it describes $\frac{16}{25}$ of the whole distance during the last second of the motion; find the whole interval occupied.

17. A point has $-m$ celos and initially it has mn velos; shew that after $(n+k)$ seconds it returns to the same point which it was passing at the end of the first $(n-k)$ seconds, with the same number of velos.

18. Prove that the distances passed over in successive equal intervals by a point starting from rest and moving with a constant acceleration are proportional to the series 1, 3, 5, 7...

19. A point having a certain velocity at a certain instant, moves with an acceleration whose direction is opposite to that of the velocity, until it comes to rest; prove that the intervals of describing the first half and second half of its path respectively are as $1 : \sqrt{2}+1$.

20. A point having α celos, passes over h feet in a certain interval, with an average velocity u velos, and its velocity is increased during the interval by v velos; prove that $uv=\alpha h$.

21. An engine-driver, whose train is travelling at the rate of 30 miles an hour, sees a danger signal at the distance of 220 yards, and does his best to stop the train; supposing that he can stop the train when travelling 30 miles an hour in 440 yards, shew that his train will reach the danger signal with a velocity of $21\frac{1}{5}$ miles per hour nearly.

22. A point, moving with a constant acceleration a celos, passed over twice as many feet in a certain interval t seconds as it did in the immediately preceding interval of t seconds; shew that its velocity at the beginning of the first interval was $\frac{1}{2}ta$ velos.

23. Draw a diagram of velocity [Art. 24] so as to exhibit pictorially the proof of Art. 23.

CHAPTER II.

MASS AND FORCE.

27. Consider some definite quantity of matter; say, a cubical lump of iron. This (i) consists of a certain *kind of material* (iron), (ii) has a definite *shape*, (iii) has a definite *volume*, (iv) contains a certain *quantity of matter*, (v) has a definite *weight*.

Each of these (i), (ii), (iii), (iv), (v) is a separate and distinct idea applicable to the lump of iron.

In dynamics we are chiefly concerned with the last two; viz., the *quantity of matter*, which we shall call the **mass**, and the *weight*.

We proceed to consider what is meant by *mass*.

28. We derive our idea of matter or mass from our muscular sense.

We also derive our idea of **force** from our muscular sense. When we exert our muscles we say we are applying *force* to something; and that to which we apply force we call *mass*.

In order to make an experiment with force we must have some means of applying a constant force to a given mass.

We know that when we compress a coiled steel spring in a certain way we have to exert a definite amount of force; also it is known that a spring can be made which will resist the same amount of force for a very considerable time.

The great difficulty in making an experiment which shall test the effect of a single force when applied to a certain mass, is the difficulty of getting rid of *all other* forces (such as friction, weight, etc.) so as to leave only the single force to be considered.

Let us imagine a perfectly smooth, perfectly horizontal sheet of ice†. Upon this, place a lump of matter, such as a smooth block of

† It must be noticed that these conditions can only be approximately fulfilled; the best ice is not perfectly smooth and the movement of the air would interfere to a certain degree with such an experiment.

stone in the form of a cube. Now take a steel spring arranged in some manner so that its state of compression can be easily observed (a spring letter-weighing machine for instance), and fastening this to the centre of one of the horizontal faces of the lump, apply a constant horizontal force by means of it to the lump, in the direction perpendicular to its face, taking care to keep the force constant and to apply it continuously and uniformly to the lump for an interval of, say, 10 seconds.

What happens? It will be found that by the force the lump has a certain amount of **acceleration** given to it; so that a velocity grows in the lump *as long as the force is applied;* and thus at the end of the 10 seconds the lump will have acquired a certain number of velos.

At the end of the 10 seconds let the force be withdrawn.

What happens? The lump has a certain number of velos; and it will be found that it will continue to move uniformly with that number of velos, so long as there is nothing in the nature of a force applied to it.

29. Suppose now that we try the experiment of Art. 28 with *several* lumps of the same size but of different material; say, one of lead, one of stone, one of cork; and suppose that by the aid of three exactly similar steel springs we can apply an equal constant horizontal force to each lump continuously for an interval of, say, 10 seconds.

What happens? It will be found that the lead, the stone, and the cork, have each a constant number of celos communicated by the force; but that the number of celos given to the cork is greater than the number given to the stone, and the number of celos given to the stone greater than the number given to the lead. So that by the end of the 10 seconds they will each have acquired a certain number of velos, the cork more than the stone, the stone more than the lead†; also, that after the force has been withdrawn, each will continue to move with its own constant velocity, so long as there is nothing in the nature of a force applied to it.

30. DEF. We choose as our **unit** *mass* the mass of a certain lump of metal [called 1 lb. (avoirdupois)].

We call this *unit mass* a **pound**, or, 1 **lb.**

† It is instructive to imagine the three blocks of the same size and shape and painted the same colour; the application of an equal horizontal force to each, will at once reveal which is lead, which stone and which cork.

31. DEF. **Force**† is that which when applied to *mass* produces in it acceleration in the direction of the force; so that the force varies as the acceleration which it produces in a given mass; and also varies as the mass in which it can produce a given acceleration [see Art. 40].

Consider (i) the experiment of Art. 28; here, if we double the force, we shall find that we produce in the same mass double the acceleration; consider (ii) the experiment of Art. 29; here, if the stone has 5 times the '*mass*' of the cork, we must have a force 5 times as large acting on the stone as on the cork to produce in each the same number of celos.

32. DEF. We choose as our **unit** *force* that force which acting on *a pound* produces in it 1 celo.

We call this *unit force* a **poundal.**

33. The statement of Art. 31 is to be given the fullest possible interpretation; it asserts that

I. *Force* is *that* which produces *acceleration* in *mass*; therefore, whenever a mass has acceleration, it is under the action of some external force.

II. When *no external force* acts on a mass it has *no acceleration*; in other words, if at rest, the mass remains at rest, and if in motion, the mass continues to move with uniform velocity.

III. When the force applied to a certain mass is doubled or trebled, etc., then the acceleration produced is doubled or trebled, etc.

IV. When the mass to which a certain force is applied is doubled, or trebled, etc., then the acceleration is halved, or divided by three, etc.

V. Every force applied to a mass produces in that mass its proper acceleration in its own direction, independently of any other motion which the mass may have.

† It is important to notice that the word **force** is in ordinary language used in more senses than one. The above definition adopts the word (*force*) as a *scientific* term and therefore limits its use to one definite idea only.

Thus, by III, Forces are measured by the number of celos which they respectively produce in 1 lb.

By IV, Masses are measured by the number of poundals required to produce 1 celo in each of them respectively.

And by V, when two equal and opposite external forces act on a particle, they produce equal and opposite accelerations; in which case the particle continues at rest, or moving with constant velocity.

34. A force which is *applied* to a mass A by another mass B is said to be **external** to A.

35. Art. 31 gives a definition of *force* in terms of *mass*; in other words, Art. 31 states the **law**† which connects the ideas of *force* and *mass*.

It will be found to combine in one statement **Newton's first and second Laws of motion** [See Chapter XI].

36. The meaning of these definitions of Force and Mass may be stated thus:

The motion of a given mass is always that of uniform velocity in a straight line, except during any interval in which it is acted on by external force.

When external force acts on a mass its motion is that of acceleration in the direction of the force, so long as the force is in action, but no longer.

The acceleration is proportional to the force which produces it; so that, when the force is changed, the acceleration is changed; when the force is withdrawn, the acceleration ceases, that is, the velocity becomes uniform.

† Considered as a definition of *force* Art. 31 assumes that we know what *mass* is. We must in fact accept the definition of the *mass* of a body as the *quantity of matter* in it.

Suppose however we take our definition of **force** as the **pressure** *which when applied to a steel spring compresses it.*

We should in this case choose as our **unit** *force*, that *force* which applied to a certain spring compresses it in a certain manner.

The statement of Art. 31 would be arranged as a definition of mass as follows.

Mass is that which, when free from the application of any force, perseveres in its state of rest or of uniform velocity; and in which the application of force produces acceleration in the direction of the force, so that the mass of a body varies inversely as the acceleration produced in it by a given force, and varies as the force required to produce a given acceleration.

We should choose as our **unit** *mass* the mass in which the unit force produces unit acceleration.

By definition, 1 poundal acting on 1 lb. produces in it 1 celo.

Therefore 1 poundal acting *continuously* for 1 second on 1 lb. adds 1 velo to its velocity.

By definition, 2 poundals acting on 1 lb. produce in it 2 celos.
and 2 poundals acting on 2 lbs. produce in it 1 celo.

Example i. *A force of* 3 poundals *is applied continuously to the mass* 1 lb. *which at the beginning of a certain interval has* 10 velos, *the force having the* same direction *as the velos; find* (i) *when the mass will have* 25 velos; (ii) how far *it will go in* 12 seconds.

By Art. 32, 3 poundals acting on 1 lb. produce in it 3 celos,
3 celos produce 15 velos in 5 seconds. (i)

The 1 lb. has 3 celos; the velocity of the 1 lb. at the middle of the 12 seconds is therefore $(10+3\times 6)$ velos; that is, 28 velos;
therefore in 12 secs. it passes over 28×12 ft., or, 336 ft. (ii)

Example ii. *A particle of* 4 lbs. *which at a certain instant has* 256 velos, *is acted on continuously by a* force *of* 32 poundals *in the direction* opposite to that of the velos; *find* (i) when *and* where *it will have* 224 velos, (ii) when and where *it will have* 224 velos in the opposite direction.

32 poundals acting on 4 lbs. produce in it $\frac{32}{4}$, that is 8, celos.
Hence the particle has -8 celos.

So that, if after x seconds the particle has 224 velos, we have

$$224=256-8x;\ \text{whence, } x=4.$$

In these 4 seconds the distance of the particle has increased by 4 times $(256-2\times 8)$ ft. or 960 feet. (i)

If after x seconds the particle has -224 velos,

we have $-224=256-8x$; whence, $x=60$.

In these 60 seconds the distance [Note p. 22] of the particle has increased by 60 times $(256-30\times 8)$ ft. or 960 ft. (ii)

Thus, the particle after 4 seconds has 224 velos; it continues moving in the same direction until the action of the force, producing in it a negative acceleration, brings it to rest; then the action of the force gradually produces in it velocity in the opposite direction, and it will be found that, as it returns, *it repasses each point with the same velocity in the opposite direction*. [See Lock's Elementary Trigonometry, chap. VIII.]

EXAMPLES. X.

NOTE. *In these and in all future examples, in which a force is applied to a mass, it is supposed that the force is so applied, that the* shape *of the mass need not be considered; in some cases, the body is supposed to be so* small *that its* size *need not be considered; such a body is called a* **particle.**

Find the acceleration produced in each of the following 6 cases.

1. by 3 poundals acting on 6 lbs.

2. by 32 poundals acting on 1 lb.

3. by 32 poundals acting on 64 lbs.

4. by 64 poundals acting on 5 lbs.

5. by 48 poundals acting on 1 cwt. [=112 lbs.]

6. by 7 poundals acting on 4 cwt.

Find the velocity produced and the distance passed over in each of the 6 following cases, the mass being *initially at rest.*

7. by 3 poundals acting on 6 lbs. for 10 seconds.

8. by 12 poundals acting on 1 lb. for 4 secs.

9. by 18 poundals acting on 27 lbs. for 3 secs.

10. by 32 poundals acting on 56 lbs. for 9 secs.

11. by 56 poundals acting on 2 cwt. for 12 secs.

12. by 64 poundals acting on 2 lbs. for $\frac{1}{2}$ sec.

13. Find the velocity acquired and the distance passed over in each of the 3 following cases, the force having the same direction as the initial velocity.

(i) 4 poundals acting for 3 secs. on 3 lbs. having initially 4 velos.
(ii) 5 poundals acting for 4 secs. on 10 lbs. having initially 20 velos.
(iii) 22 poundals acting for $3\frac{1}{2}$ secs. on $\frac{1}{2}$ lb. having initially 4 velos.

14. 20 poundals act on a mass of 5 lbs. (initially at rest) for 3 seconds and then cease; how far will the mass go in 10 seconds more?

15. 64 poundals act for 4 seconds on a mass of 2 lbs. initially at rest; after which the force is suddenly reversed; find how far the mass goes in 8 seconds from rest.

16. A particle of 1 lb. which at a certain instant has 96 velos, is acted on by 32 poundals in the direction *opposite* to the velos; when will it be 128 ft. from its position at that instant? and find its velocities at those times.

17. A particle of 1 lb. which at a certain instant has 64 velos, is acted on by 32 poundals in the direction opposite to the velos; when will it be at rest? when will it be moving in the opposite direction with 32 velos?

18. A particle of 1 lb. acted on by a constant force moves in a certain second over 20 ft., and in the next second but one it moves over 128 ft.; find the force.

19. A particle of 1 lb. is at rest, when being acted on by a constant force, it moves over 16 ft. in the first second of its motion; find the force.

20. A mass of 20 lbs. is acted on by a constant force, and in the fifth and seventh seconds of its motion it passes over 108 ft. and 140 ft. respectively: find its initial velocity and the magnitude of the force.

21. A particle acted on by a constant force of 20 poundals passes over 72 ft. while its velocity increases from 16 to 20 velos; find its mass.

22. A particle acted on by 20 poundals attains a velocity of 45 miles per hour after 12 seconds from rest; find its mass.

23. A train of 90 tons on a horizontal plane at rest is acted on by a force of 1000 poundals; the force acts for one minute and then ceases; find how far the train travels before coming again to rest, supposing the friction of the rails etc. to be equivalent to a retarding force of 300 poundals. [N.B. 1 ton = 2240 lbs.]

24. A train of 100 tons attains from rest a velocity of 40 miles per hour in going 1 mile; the breaks are then applied and the train comes to rest in 440 yds. Compare the retarding force exerted by the breaks with the force exerted by the engine, (neglecting the rotary motion, and the ordinary friction of the wheels etc.).

25. A force of p poundals acts for n minutes on m lbs.; find the velocity generated.

26. Find how many poundals will produce in 10 lbs. a velocity of 30 velos after going 25 yards from rest.

27. A train of 100 tons is impelled along a horizontal railroad by a horizontal force of 23100 poundals; neglecting friction etc., how long will it take to go 4 miles from rest? and how many miles per hour will it be going at the end of the 4 miles?

28. A mass of 10 lbs. acted on by a constant force is at a certain instant observed to have 20 velos and is observed to go twice as far in the third and fourth seconds reckoned from that instant as it does in the first two seconds; what is the force?

29. Two masses A, B of 1 lb. each on a smooth horizontal plane are at a certain instant at the same place, A has 4 velos, B is at rest; if B is acted on by 2 poundals in the direction of B's velocity when will it overtake A?

30. A mass of 4 lbs. acted on by 16 poundals is observed to have an average velocity of 100 velos for 10 seconds; what was its velocity at the beginning of those 10 secs.?

37. The force which produces a celos in m lbs. is ma poundals.

For, to produce 1 celo in mass 1 lb. 1 poundal is required;
to produce 1 celo in mass m lbs. m poundals are required;
to produce a celos in mass m lbs. ma poundals are required.

Example. A force f poundals *acts continuously on a particle* m lbs. *which at a certain instant has* u velos *in the direction of the force; find* (i) *the* acceleration, (ii) *the* velocity *after* t seconds, (iii) *the* distance *it passes over in* t seconds, (iv) *how many* seconds *it takes to go* s feet.

1 poundal acting on 1 lb. produces 1 celo.

f poundals acting on 1 lb. produces f celos.

f poundals acting on m lbs. produce $\underline{\dfrac{f}{m}}$ celos. (i)

In t seconds $\dfrac{f}{m}$ celos *produce* $\dfrac{ft}{m}$ velos, therefore after t seconds the velocity is

$$\underline{\left(u+\frac{ft}{m}\right) \text{ velos.}} \qquad \text{(ii)}$$

The velocity at the middle of the interval of t secs. is $\left(u+\frac{1}{2}\dfrac{ft}{m}\right)$ velos, and therefore the distance passed over in t seconds is

$$\underline{\left(ut+\frac{1}{2}\frac{ft^2}{m}\right) \text{ feet.}} \qquad \text{(iii)}$$

Let the particle take x seconds to pass over s feet, then by (iii)

$$s=ux+\frac{1}{2}\frac{fx^2}{m},$$

from which quadratic equation x may be found. (iv)

EXAMPLES. XI.

1. How many poundals are required to produce g celos in m lbs.?

2. How far will a ton [2240 lbs.] acted on by f poundals go from rest in t seconds?

3. How many poundals are required to produce g celos in k tons?

4. 1000 poundals acts upon a train of k tons on a horizontal plane; the friction of the rails etc. is equivalent to a retarding force of 250 poundals, how long will the train take to get up from rest a speed of m miles per hour?

38. DEF. The **mass-velocity** (or, the **momentum**) of a particle is that which varies as the mass and also as the velocity of the particle.

We shall choose as our **unit** *mass-velocity* that of a particle of 1 lb. moving with 1 velo.

We shall call this unit a **pound-velo**.

Hence, the *number* of *pound-velos* which a particle has, is the product of the *number* of lbs. in it by the *number* of velos with which it is moving.

39. DEF. The **mass-acceleration** of a particle is that which varies as the mass and also as the acceleration of the particle.

We shall choose as our **unit** *mass-acceleration* that of a particle of 1 lb. moving with 1 celo.

We shall call this unit a **pound-celo**.

Hence, the *number* of *pound-celos* which a particle has, is the product of the *number* of lbs. in it by the *number* of celos with which it is moving.

It should be noticed that the **measure** of the mass-acceleration of a particle is the **rate** of increase of its mass-velocity.

Example i. *Find the* mass-velocity *of a train of* 120 tons *going* 30 miles an hour.

$$120 \text{ tons} = 120 \times 20 \times 112 \text{ lbs.}$$

$$30 \text{ miles an hour} = 44 \text{ velos.}$$

Therefore, the mass-velocity of the train is

$120 \times 20 \times 112 \times 44$ pound-velos, that is 11,827,200 pound-velos.

Example ii. *Find the* mass-acceleration *of a* man and bicycle *of* 20 stone *while attaining a speed of* 15 miles an hour *in going* 121 ft. *from rest.* [1 stone = 14 lbs.]

$$15 \text{ miles an hour} = 22 \text{ velos.} \qquad \text{[Ex. ii. p. 3.]}$$

We have $$2as = v^2,$$

or, $$2a \times 121 = (22)^2; \text{ therefore, } a = 2.$$

Also $$20 \text{ stone} = 20 \times 14 \text{ lbs.}$$

Therefore, the required mass-acceleration is 560 pound-celos.

EXAMPLES. XII.

Find the mass-velocity (or, momentum) of each of the following:

1. A mass of 3 lbs. having 3 velos.

2. A mass of 1 cwt. having 10 velos.

3. A train of 100 tons going at the rate of 30 miles an hour.

4. A bullet of 1 ounce going 1000 ft. per second.

5. A cannon ball of 1 cwt. going 1000 yds. per second.

6. A man of 12 stone running 12 miles an hour. [A stone = 14 lbs.]

Find the mass-acceleration of each of the following:

7. A mass of 10 lbs. having 8 celos.

8. A mass of 2 cwt. having 32 celos.

9. A train of 100 tons which has gained from rest a velocity of 30 miles per hour in 1 minute.

10. A bullet of 1 oz. which in traversing 36 inches from rest has gained a velocity of 1000 ft. per second. [16 oz. = 1 lb.]

11. A cannon ball of 56 lbs. which in traversing the tube of a cannon 10 ft. long, from rest, has gained a velocity of 500 yards per second.

12. A skater of 12 stone, who starting from rest has acquired a velocity of 20 miles an hour in going 50 yds.

40. It follows from Art. 31 that the number of poundals in an external force acting on a mass, is equal to the number of pound-celos which it produces in that mass.

Hence Art. 31 may be stated thus:

DEF. **Force** is numerically equal to and in the same direction as the mass-acceleration which it produces.

Example i. *A force* 32 poundals *is applied to a ton;* find (i) *the* acceleration *produced,* (ii) how long *this force will take to produce* 6 velos *in the ton.*

Let the number of celos produced by the 32 poundals be a; then, the mass acceleration produced is $2240a$ pound-celos.

By Art. 40 the external force which produces $2240a$ pound-celos is

$$2240a \text{ poundals.}$$

But, this force is $\qquad 32$ poundals.

Therefore, $\qquad 2240a$ poundals $= 32$ poundals;

that is, $\qquad 2240a = 32$; or, $a = \frac{1}{70}$.

Thus, the acceleration produced is $\frac{1}{70}$ of a celo. (i)

$\frac{1}{70}$ celo in 1 second produces $\frac{1}{70}$ velo,

$\frac{1}{70}$ celo in 6×70 seconds produces 6 velos.

Therefore, if the force be continuously applied to the ton it will produce 6 velos in 7 minutes. (ii)

Example ii. *A cannon ball of* 250 lbs. *starting from rest, leaves the mouth of a cannon with a velocity of* 1000 ft. per second; *find the resultant force which acted upon it when in the cannon, supposing this force to be uniform and that the ball passed over* 20 *ft. while in the cannon.*

[NOTE. By *resultant* force is meant, the force over and above that necessary to overcome the friction of the tube; the pressure of the gases on the base of the shot is necessarily greater than this resultant force, which is only that part of this pressure which produces the acceleration of the shot.]

To find the number of celos which the ball had in the cannon, we have $v = 1000$, $s = 20$ ft., $u = 0$, and $2as = v^2$.

Therefore, $40a = 1000000$; or, $a = 25000$.

The mass-acceleration is ma;

that is, 25000×250, or 6250000 pound-celos.

The number of poundals which produce 6250000 pound celos = the number of pound-celos produced; that is, = 6250000 poundals.

EXAMPLES. XIII.

1. Find the number of poundals required to produce the motion in each of the examples 7 to 12 of Examples XII.

2. Find the number of poundals which by acting for 20 secs. produce the mass-velocities in Questions 1 to 6 of Ex. XII. respectively.

3. A locomotive engine can exert upon a train of 100 tons a force of 10000 poundals more than is necessary to counteract the friction; how long would this engine take in working the train on a level line up to a speed of 30 miles an hour?

4. A rifle bullet of 1 oz. leaves the barrel of a rifle which is 2 ft. 6 in. long with a velocity of 1200 feet per second; find the resultant force, supposed uniform, with which it was acted on in the barrel.

5. A mass of 10 lbs. moves over 16 ft. from rest in 2 seconds; supposing its acceleration to be uniform, what force is acting on it?

6. It is observed that 1 ton is moving in a straight line with a velocity of 10 miles per hour; 3 minutes later it is observed to be moving in the same straight line with a velocity of 12 miles per hour; supposing a uniform external force to have acted on the mass during those 3 minutes, how many poundals did it contain?

WEIGHT.

41. DEF. The **weight** of a mass is the *force* with which it is attracted by the earth.

When we support a mass in our hand so as to keep it at rest, we can only do so by the application of an upward pressure or force; this upward *force* is equal and opposite to the *weight* of the mass.

42. Since the weight of a mass is a force, and since our unit of force is a poundal, in order to measure the weight of a given mass we must express its weight in poundals.

The number of poundals in any force is equal to the number of pound-celos which it produces; accordingly we have to ascertain how many celos the weight of any particular mass produces in that mass.

43. The following experiment is important.

A chamber, such as the receiver of an air-pump, is exhausted of air. In the chamber are placed several masses of different kinds and of different sizes; for instance, a leaden bullet, an iron nail, a bronze penny, a gold sovereign, a glass ball, a piece of cork, a feather; these are all placed on a ledge so arranged that they can be simultaneously let fall.

When a mass is 'let fall' in a vacuum, the *only* force acting upon the mass is its own weight; accordingly in this experiment when the masses are let fall, the weight of each mass produces in its own mass motion, in a vertical direction downwards, which is that of uniformly increasing velocity.

Now it is observed that, having been simultaneously let fall from the same height, these different masses all reach the bottom of the chamber *simultaneously*.

But, when a number of points, starting from rest and each moving in the same direction with uniformly increasing velocity, pass over equal distances in the same interval of time, they must all be moving with the *same acceleration*. Hence,

44. The *weight* of a mass is a *force*, which produces in its own mass a certain definite acceleration; which acceleration is the same at the same place on the earth's surface for

all masses of whatever magnitude and of whatever kind of material.

This definite acceleration, which is called the **acceleration due to gravity**, is different at different places on the earth's surface.

It is greatest near the Poles, (where it is about 32·255... celos) and it gradually diminishes to about 32·091... celos at the equator.

It is the same at places having the same latitude at the same elevation above the level of the sea.

It has been ascertained by observation that at Greenwich the number of celos in any mass when *falling freely* towards the earth's surface (that is, when moving either *upwards* or *downwards* under the action only of its own weight) is 32·19....

The weight of 1 lb. produces 32·19... pound-celos.

Hence **the weight of 1 lb. = 32·19... poundals.** [Art. 40.]

45. It is usual to use the letter g to indicate the *number* of units of acceleration produced in a mass by the action of its own weight.

Hence, at Greenwich $g = 32·19....$

The mass-acceleration produced in m lbs. by the *weight* of m lbs. is mg pound-celos. [$m \times (32·19...)$ pound-celos.]

The *force* required to produce mg pound-celos is mg poundals. Therefore

the weight of m lbs. is the *force* **mg poundals.**

N.B. In all numerical examples we shall use
32 poundals as a sufficient approximation to
the **weight** of 1 lb.

Example i. *How many* poundals *are there in* the weight of 3 lbs.?

The weight of 3 lbs. produces in the mass 3 lbs. (about) 32 celos.

Therefore the *weight* of 3 lbs. is equivalent to the *force* 3 × 32 poundals.

Example ii. *How many* celos *will a force equal to* the weight of 3 lbs. *when applied to a* mass 12 lbs. *produce in it?*

The force is 3 × 32, or, 96 poundals
and 96 poundals produces in 12 lbs., $\frac{96}{12}$ or 8 celos.

Example iii. 16 lbs. *is acted on continuously by a force equal to the weight of* 4 lbs.; *find its* acceleration; *and find how far it moves from rest in* 10 seconds.

The weight of 4 lbs. $=4\times 32$ poundals.

This force 4×32 poundals acting on 16 lbs. produces in it $\frac{4\times 32}{16}$ celos;

That is, this force produces in 16 lbs. the acceleration 8 celos.

In 10 seconds the 16 lbs. will pass over a distance s feet such that

$$s=\tfrac{1}{2}at^2=\tfrac{1}{2}\times 8\times 100=400.$$

The distance required is 400 feet.

NOTE. A force is said to be expressed *in pounds weight*, when we state the number of lbs. whose *weight* is equal to the force.

Example iv. A mass 4 cwt. *is observed to be moving with* 16 celos; *express* in pounds weight *the* force *which is acting upon it.*

When 4 cwt. has 16 celos, it has $4\times 112\times 16$ pound-celos.

To produce this, $4\times 112\times 16$ poundals must be acting upon it; $4\times 112\times 16$ poundals are equal to the weight of $\frac{4\times 112\times 16}{32}$ lbs.; that is, the weight of 2 cwt. must be acting on the 4 cwt.

EXAMPLES. XIV. a.

Express the four following forces in poundals:

1. The weight of 3 lbs. **2.** The weight of half an ounce.

3. The weight of 20 tons. **4.** The weight of 4 cwt.

Express the six following forces in lbs. weight:

5. 128 poundals. **6.** 1000 poundals.

7. The force which produces 32 celos in 100 lbs.

8. The force which produces 4 celos in 1 cwt.

9. The force which acting on 1000 lbs. will produce in it 20 velos in 5 secs. from rest.

10. The force which is acting on a mass of 4 cwt. which moves over 64 ft. from rest in 2 seconds.

11. 1 cwt. placed on a smooth horizontal plane is acted on continuously by a horizontal force equal to the weight of 7 lbs.; find its acceleration.

12. 3 tons placed on a smooth horizontal plane is acted on continuously by a horizontal force of 42 lbs. weight; how long will it take to gain from rest a velocity of 15 miles an hour?

13. 28 lbs. placed on perfectly smooth ice is acted on by a horizontal force equivalent to the weight of 1 lb.; find its velocity after 7 minutes.

14. A train of 100 tons is drawn with an acceleration of 4 celos; express in pounds weight the force exerted by the engine in addition to that necessary to overcome the friction.

15. 1 ton on a smooth horizontal plane is observed to move from rest a distance of 48 ft. in 60 seconds; if the motion is caused by a uniform horizontal force, how many lbs. would the force lift?

16. A train of 100 tons is drawn by an engine which can exert a force, which would lift a weight of 800 lbs.; supposing the force of the friction etc. to be equivalent to a force of 100 lbs. weight, find how long the train would be in attaining from rest a velocity of 60 miles per hour.

17. A mass of 1 cwt. is placed on ice and is acted on by a horizontal force equal to the weight of 8 lbs.; supposing that a force equal to the weight of 1 lb. is sufficient to overcome the friction of the ice, find when the mass will have 100 velos.

18. Supposing the force of 8 lbs. weight of Question **17** to be withdrawn when the mass is moving at the rate of 60 miles an hour, in how many seconds will the mass be brought to rest by the retardation due to the friction?

19. A train is acted on by a horizontal force (over and above that necessary to overcome the friction) equal to the weight of 10 cwt.; it is observed to pass over 20 ft. from rest in 30 seconds; find the mass of the train.

20. A lump of chalk is placed at rest on a smooth horizontal plane and is acted on continuously for 10 seconds by a force equivalent to the weight of 2 lbs.; the force is then withdrawn and after that the mass is observed to move 100 yards in half a minute. What is the mass of the chalk?

21. Two masses consisting of a cubic foot of lead and a cubic foot of iron placed at rest on perfectly smooth ice are each acted on in the same direction by a horizontal force equal to the weight of 1 lb.; how far will they be apart after 30 seconds?

[NOTE. A cubic ft. of lead weighs about 11340 oz.; a cubic ft. of iron weighs about 7680 oz.]

22. A mass of 10 lbs. under the action of gravity is acted on also by a continuous vertical force upwards equivalent to the weight of 3 lbs.; find how long the mass will take to fall 4480 ft. from rest.

23. A mass of 1 cwt. slides down a vertical tube so that the retardation of friction is equivalent to the weight of 12 lbs.; how long will this mass take to fall 112 ft. from rest under the action of gravity?

24. A mass of 20 lbs. is observed to fall under the action of gravity with an acceleration of 24 celos; what other force is acting upon it besides the attraction of the earth?

25. A mass under the action of gravity is also acted on by a force vertically downwards equivalent to the weight of 4 lbs. The mass is observed to fall 128 ft. in the first two seconds of its motion. What is the mass?

26. A train on a horizontal plane is observed to be moving at a certain instant at the rate of 15 miles per hour; 3 minutes 40 secs. later it is moving at the rate of 30 miles per hour. The force exerted by the engine is equivalent to the weight of 10 cwt. and the retardation of friction to the weight of 6 cwt.; find the mass of the train.

46. The motion of a mass falling vertically (either *upwards* or *downwards*) under the action of its own weight only is that of a point moving with 32 celos downwards.

Example†. *How long does a stone take in falling from rest under the action of gravity through a vertical distance of* 128 ft.?

By Arts. 43—45 the stone has an acceleration of 32 celos downwards.

Let t seconds be the time of passing over 128 ft.;

[Since $s = ut + \frac{1}{2}at^2$, and $ut = 0$]

then, $128 = \frac{1}{2} \times 32t^2$;

or, $t^2 = 8$;

therefore, $t = 2\sqrt{2} = 2 \times 1\cdot4142 = 2\cdot8284\ldots$

The interval occupied is therefore 2·8... seconds.

EXAMPLES. XIV. b.

1. A stone is let fall from the top of a tower 256 ft. high; in how many seconds will it reach the ground?

2. A stone is let fall; how far will it fall (i) in the 2nd second, (ii) in the 4th second from rest?

3. A stone is thrown vertically upwards with a velocity of 320 velos; when will it come to rest? and how high will it go?

4. A stone is thrown vertically upwards and returns after 4 seconds to the point from which it was thrown; with what velocity was it thrown? and how high did it go?

5. A stone is thrown vertically upwards and passes a point 48 ft. high with a velocity of 32 ft. per second; in how many seconds will it return to the point of projection?

6. A stone is let fall from a tower 256 ft. high and at the same instant another stone is projected from the foot of the tower with just sufficient velocity to carry it to the top of the tower; when and where will the stones meet?

† Example iv. p. 16 and Example i. p. 22 should be noticed.

47. In Art. 28 it is assumed that we can apply a *constant* force to a mass.

Such a force might be obtained by taking care that the force is just sufficient to keep a certain steel spring compressed in a certain manner.

By applying the same constant force to each of two bodies in turn and observing the number of celos produced in them, we obtain a measure of the mass of the bodies.

It happens however that the experiment of Art. 43 provides a more convenient method of measuring the mass of a body. For, since the weight of a mass is a force which produces a fixed number (g) of celos in the mass itself, the *weight* of a mass is proportional to the mass itself; therefore, **equal masses have equal weights.**

By a *ton* of coal is meant a certain amount of mass.

That a certain quantity of coal contains a ton mass is ascertained by weighing it; that is, we ascertain that the mass of coal has the same weight as a certain other standard mass of iron, which mass we call a ton.

In order to understand what is meant by the statement '*equal masses have equal weights*', let us consider what would be the practical result if two equal masses had not equal weights.

Imagine for example that we have two lumps, one of iron, the other of brass, which are of equal mass but have very different weights. That is, suppose the Earth attracts one of the masses, say the iron, very strongly, the brass very slightly.

The *masses* are equal; let them be placed on a smooth horizontal plane; apply equal horizontal forces with a steel spring or with the hand; they will each have the same acceleration. Bring them to rest again and lift them vertically; the iron would require a much larger force, to support it when lifted, than the brass. Or again, place them in the opposite pans of a weighing machine; then one of the pans would be depressed with a strong force.

NOTE. In the language in common use, the ideas of *mass* and *weight* are not clearly distinguished.

For example, a piece of metal is said to be

'*a* 7 lbs. *weight*';

these words mean, a piece of metal which is '*a mass of* 7 lbs. *whose* weight *is used for the purpose of comparing the* weights *of other masses*.'

Or again, 'a train *weighing* 100 tons';

here, the word *weighing* means, '*whose* weight *is equal to that of*' 100 tons.

Again, a string is often said to be *light*, meaning *weightless*, or *of too small* **weight** *to be observed;* such a string must also *be of too small* **mass** *to be observed.*

The student however must carefully remember that

the **unit mass** is 1 lb.,

while **the weight of 1 lb. is 32·2....poundals.**

48. The two following examples should be noticed and the results compared.

Example i. *A mass of* 1 ton *lies on a smooth horizontal plane; a man by means of a windlass and string applies continuously to the mass a horizontal force equal to the weight of* 28 lbs. [i. e. keeps the tension of the string always equal to the force of 28 lbs. weight]; *how long will he take to move the mass* 5 ft. *from rest?*

Let the force produce α celos in the ton; that is, in 2240 lbs.

The mass acceleration is therefore 2240α pound-celos.

To produce 2240α pound-celos the force required is

2240α poundals.

The force actually applied is the weight of 28 lbs.;

which is, 28×32 poundals.

Hence 28×32 poundals $= 2240\alpha$ poundals;

whence, $\alpha = \frac{2}{5}$.

Let s ft. be the distance moved from rest in time t seconds; then,

$$s = \tfrac{1}{2}\alpha t^2;$$

here, $5 = \frac{1}{2} \times \frac{2}{5} \times t^2$;

or, $t^2 = 25$; therefore, $t = 5$.

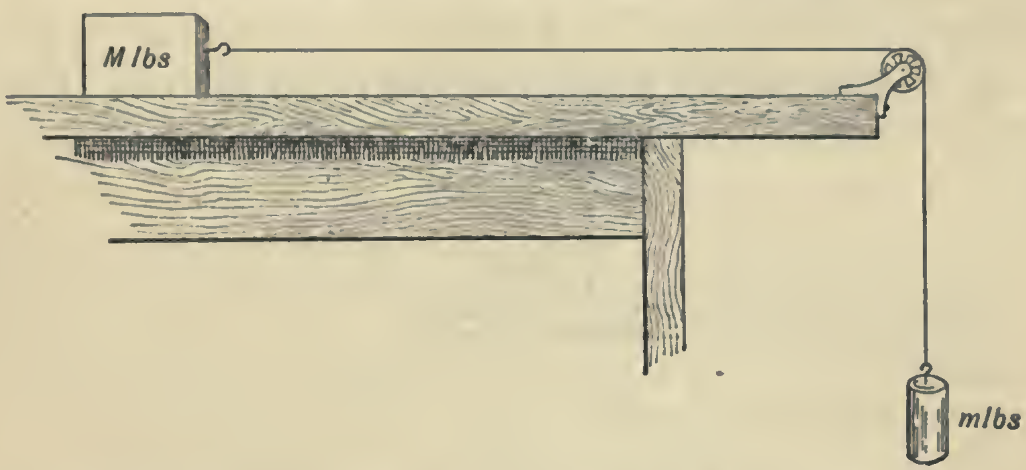
The time required is 5 seconds.

Example ii. *A mass of* 1 ton *lies on a smooth horizontal table, and is attached by a light string which passes over the edge of the table, to a mass of* 28 lbs., *which hangs freely under the action of its own* weight; *how long will the masses take to move* 5 ft. *from rest? also find the tension of the string.*

[NOTE: Here, the force acting on the ton mass is **not** equal to the weight of 28 lbs.; for

The force acting on the ton is the pull or *tension* of the string; and the tension of the string is *not* equal to the weight of the 28 lbs.; for if it were, the 28 lbs. mass would have acting on it (i) a pull equal to the weight of 28 lbs. *upwards* and (ii) its own weight (which is that of 28 lbs.) *downwards;* that is, it would have no *resultant* external force acting on it, and would therefore have *no acceleration.*

But, if the ton has acceleration, the 28 lbs. must have acceleration also. Hence, the force acting on the ton, i.e. the tension of the string, is *less* than the weight of 28 lbs.; as will presently appear.]



I. Consider the external forces acting on the 28 lbs.†

Let T poundals be the tension of the string; then there are two external forces acting on the mass of 28 lbs., its weight directly downwards, and the pull of the string directly upwards,

$$\therefore \text{ the resultant force is } 28 \times 32 \text{ poundals} - T \text{ poundals.}$$

Let this resultant force produce a celos in the 28 lbs.

It produces therefore $28a$ pound-celos;

the force necessary to do this is $28a$ poundals; hence we see that

$$(28 \times 32 - T) \text{ poundals} = 28a \text{ poundals} \qquad \text{(i).}$$

† The student must learn to fix his attention exclusively on *one mass only*, forgetting for the time the existence of the other mass.

II. Consider the external force acting on the ton.

The *weight* of the ton does not concern the question; it is supposed to be entirely obliterated by the upward support of the smooth horizontal plane.

The only other force acting on the ton is the tension of the string. The string is supposed to be *light*, and therefore of no mass; so that it exerts the same pull on the ton as it does on the 28 lbs.; which pull is

T poundals.

Also, since the two masses are connected by an *inextensible* string, the acceleration of the ton is the same as that of the 28 lbs. viz., a celos.

The force necessary to produce a celos in 2240 lbs. is

$$2240\,a \text{ poundals};$$

but, the force which produces the a celos is T poundals; hence we see that $\quad T \text{ poundals} = 20 \times 112a \text{ poundals} = 80 \times 28a \text{ poundals}; \qquad$ (ii)

but from I, $\quad (28 \times 32 - T) \text{ poundals} = 28a \text{ poundals}; \qquad$ (i)

whence, adding (ii) to (i) $\quad 28 \times 32 = 81 \times 28a;$

therefore, $\quad a = \frac{32}{81}.$

Let t seconds be the time in which the masses pass over 5 ft. from rest;

then, $\quad 5 = \frac{1}{2} \times \frac{32}{81} t^2;$

or, $\quad t = \frac{9}{4}\sqrt{5} = \frac{9}{4} \times 2{\cdot}236\ldots$

$\quad = 5{\cdot}031\ldots$

The time required is 5·031... seconds.

Also, $\quad T = 20 \times 112a = 20 \times 112 \times \frac{32}{81},$

$\therefore$ the tension = the weight of $20 \times 112 \times \frac{32}{81} \times \frac{1}{32}$ lbs.

= weight of $27\frac{53}{81}$ lbs.

NOTE. Comparing these two *Examples* we observe that in *Example* i. the acceleration of the ton is $\frac{32}{80}$ celos, in *Example* ii. the acceleration is *less*, viz. $\frac{32}{81}$ celos. In *Example* i. the weight of 28 lbs. acts on a ton, in *Example* ii. the weight of 28 lbs. acts on (a ton + 28 lbs.).

49. DEF. The **inertia** or **inertness** of *mass* is that essential property by virtue of which mass requires the application of a finite external force for a definite interval of time in order to produce a finite change in its velocity.

Consequently it is by virtue of its inertia that mass per-

sistently retains its existing velocity unaltered, so long as it is not acted on by external force.

The nature of *force* is such that it needs *time* in which to change the velocity of mass.

Mass is **inert**, in the sense that it cannot of itself make any *change* in its velocity.

External force acting on mass produces in it acceleration only; it does *not* produce any definite *velocity* unless it acts *continuously* for some definite interval of time.

Example. When it is observed that a given mass M has moved in a given interval from a point A to another B, we can from this infer nothing as to whether an external force has acted on the mass during that interval or not. The mass M may have moved over that distance by reason of its inertia; by virtue of which it persisted in moving from A to B with uniform velocity.

But when it is observed that a mass M has at one instant a certain *velocity* and that after some interval it has a different velocity, we are able to state that, for some part at least of that interval, it has been acted on by some external force.

In the Examples of Art. 48, the reason why the acceleration of the ton in Example i. is greater than that of the ton in Example ii. is that the weight of 28 lbs. in Ex. i. has to overcome the '*inertia*' of 1 ton only while in Example ii. it has to overcome the *inertia* of (1 ton + 28 lbs.).

EXAMPLES. XV.

1. A sledge of 1 ton is pulled on ice by a rope whose tension is equal to the weight of 56 lbs.; if the friction of the ice causes a horizontal retarding force on the sledge equal to $\frac{1}{80}$ of its weight, find the acceleration produced.

2. A sledge of 1 ton is pulled on ice by 4 dogs whose weight is 28 lbs. each; each dog obtains from the friction of the ice a horizontal force forwards equivalent to half its own weight; prove that if the friction of the sledge on the ice is $\frac{1}{80}$ of its weight the acceleration produced is $\frac{16}{105}$ celos.

3. A man of 12 stone on a railway truck weighing $1\frac{1}{2}$ tons propels it on a smooth horizontal railway, when he is himself on it, by means of a rope fastened to a fixed point; he pulls with a constant force of 40 lbs. weight; compare the acceleration of the truck with that which he would produce if, fastening the rope to the truck and getting off, he pulled at the rope with a force of 40 lbs. weight.

4. A mass of $1\frac{1}{2}$ tons placed on a smooth horizontal table is acted on by a horizontal force of 40 lbs. weight; find how far it would move from rest in 2 seconds.

5. A mass of $1\frac{1}{2}$ tons lies on a smooth horizontal table and is attached by a light string, which passes over the edge of the table, to a mass of 28 lbs. which hangs freely under the action of its own weight; how far will the masses move from rest in 2 seconds?

6. A man of 12 stone finds that he can in running attain a speed of 15 miles an hour in 3 seconds; find the horizontal force which he applies to his own mass, supposing that he attains that speed with uniform acceleration.

7. The man in Question **6** ties a string to a mass of 5 cwt. on a smooth horizontal sheet of ice and starts to run on the bank, also horizontal, using exactly the same horizontal force as in **6**; how long will he take to attain a speed of 15 miles an hour?

8. A train of 240 tons running at the rate of 30 miles an hour on a straight horizontal line comes to rest, when under the action of friction only, after running 4 miles; assuming the force of friction to be constant, find the force in lbs. weight which would just cause the train to move when at rest.

9. Express in lbs. weight the force which the engine of the train of Question **8** must exert that it may attain a velocity of 60 miles an hour in 1 min. 28 secs. from rest.

10. A train consisting of 10 trucks of 20 tons each is pulled by an engine of 50 tons on a smooth horizontal railway with an acceleration of $\frac{1}{20}$ celo; find (i) the force which the engine obtains from the friction between its driving wheels and the rails, (ii) the force it exerts on the train of trucks.

11. In Question **10**, supposing the engine to be changed for one of 80 tons, then if the acceleration is the same as before, in what way will the forces (i) and (ii) be altered?

12. An engine of 50 tons can exert a force (derived from the friction between its driving wheels and the rails) sufficient to draw a train of 10 trucks of 20 tons each with $\frac{1}{25}$ celo; with what acceleration could it draw a train of 15 trucks of 20 tons each?

CHAPTER III.

STRESS.

50. When two **equal forces** act *simultaneously* throughout some interval each on one of two masses M and M', they produce in each mass the *same number of pound-velos.*

For at any instant one force is producing in the mass on which it acts the same number of pound-celos which the other force is at that instant producing in the mass on which it acts. Therefore in the same interval each force produces the same number of pound-velos.

The forces are not necessarily *uniform*, provided they are always *equal;* one force may be changed provided the other force is simultaneously changed also.

When the *masses* M and M' are not equal, the *accelerations* produced in them by these equal forces will be unequal; and therefore the *velocities* added to the masses respectively in the course of any interval will be unequal.

51. It happens that the action of two exactly equal forces, acting simultaneously for the same interval of time on different masses, in exactly opposite directions, is a phenomenon which is constantly taking place.

52. No external force can be applied to one mass by another, without the first mass exerting an equal and opposite force on the mass which applies the force. This is

Newton's Third Law. '*To every action there corresponds an equal and opposite reaction.*' [See Chapter XI.]

53. DEF. **A stress** is that which consists of two *equal* and *opposite* forces together forming an action and its reaction.

Examples. The *pressure* of a heavy mass on the ground on which it rests, and the pressure of the ground on the mass, together form a stress.

The *tension* of a string at any point of its length is a stress.

54. When two particles act and react the one on the other, the stress causes no alteration in the *total* mass-velocity of the two particles; that is, the stress has no effect on the algebraical *sum* of their mass-velocities.

For a stress consists of two equal and opposite forces acting simultaneously; so that [Art. 50] whatever amount of mass-velocity the action produces in one mass in any interval, the reaction produces in the other mass in the same interval an exactly equal amount in the opposite direction.

55. When the two masses m lbs. and m' lbs. are so connected that their velocities are always equal, since the stress set up between them has no effect on the sum of their mass velocities, [that is, upon $(m+m') \times v$,] it follows that the stress has no effect [upon v, that is,] upon their velocity.

DEF. When two particles are considered as a single mass the stress between them is said to be **internal**.

Hence the above result may be stated thus,

An *internal stress* between two particles (which cannot move relatively to each other) *has no effect on their joint velocity.*

Example i. The stress communicated by a light inextensible string from a mass m lbs. to another mass m' lbs., has no effect in altering their joint velocity. Provided the string is always kept tight they may be treated as one mass, and the tension of the string need not be considered in discussing their joint velocity.

Example ii. *Two masses* m lbs. *and* m′ lbs. *placed on a smooth horizontal plane are connected by a light inextensible string; a force of* f poundals *is applied to the mass* m *in the direction of the string, which is tight; find the* acceleration *produced; also the* tension *of the string.*

I. The masses are prevented by the string from moving relatively to each other; hence, we may consider the tension of the string as an internal stress, and consequently the two masses as a single mass.

Thus, we have a mass of $(m+m')$ lbs. acted on by f poundals; therefore the acceleration α is given by the equation $f=(m+m')\,\alpha$;

that is, the acceleration is $\dfrac{f}{m+m'}$, celos (i).

II. To find the *tension* of the string we must consider each component of the stress separately.

Let the string apply an external force $-T$ poundals to the mass m; it will consequently apply an external force T poundals to the mass m'.

Then, if a celos be the acceleration (which is the same for each mass, for the string is tight and inextensible)

we have, $f - T = ma$; [from the motion of m

and, $T = m'a$; [from the motion of m

whence, by addition, $f = (m + m')\,a$ (i),

wherefore $T = \dfrac{m'}{m + m'} f.$

The tension required is $\dfrac{m'}{m + m'} f$ poundals (ii).

EXAMPLES. XVI.

1. An engine of 40 tons is attached to a train of 6 carriages of 10 tons each; the train is moving with 5 celos; neglecting friction and the rotatory motion of the wheels, find the tension of the coupling between the engine and the next carriage.

2. In the train in Question 1 find the tension of the coupling between each pair of carriages.

3. In the train in Question 1, what force is the engine exerting?

4. The train in Question 1 moving at the rate of 30 miles an hour is stopped by brakes on the engine in a quarter of a mile; find

(i) the force exerted by the brake,

(ii) the stress between the engine and the next carriage,

(iii) the stress between the two last carriages,

(iv) if the brakes on the engine exerted the same force in all cases, how far would the engine alone, going 30 miles an hour when the brakes are applied, run before coming to rest?

5. An engine which exerts a horizontal force sufficient to generate an acceleration a celos in its own mass m lbs., is attached to 3 carriages whose masses are $\frac{1}{2}m$ lbs., $\frac{1}{4}m$ lbs. and $\frac{1}{8}m$ lbs. respectively; find the acceleration of the train and the stress between each carriage, neglecting friction and the rotary motion of the wheels.

6. An engine which exerts a horizontal force p poundals, sufficient to generate in its own mass a celos, is attached to n carriages each of which is $\frac{1}{10}$th of the mass of the engine; find the acceleration of the train and the stress between the 4th and 5th carriages, neglecting friction etc.

56. The laws of motion may be illustrated and tested experimentally by the aid of **Atwood's machine**.

Atwood's machine may be described as follows.

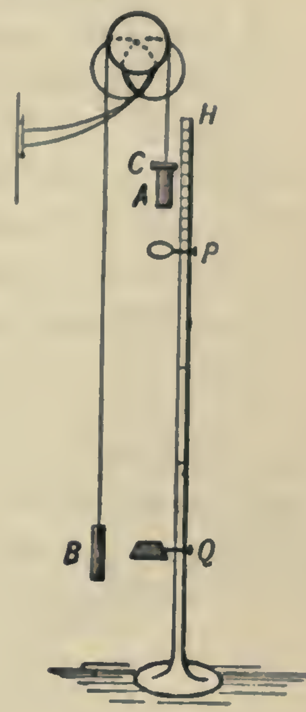
A wheel or pulley, made as light and as free from friction as possible, is fixed at a convenient height from the ground; over the pulley is passed a light inextensible string, usually made of silk; from the ends of the string are suspended two masses M and M'. Each mass is acted on only by its own weight and by the tension of the string.

I. It will be found that we can arrange the masses so that they move with a very *small* acceleration compared with that produced in a mass falling freely under the action of its own weight; so that we can ascertain by actual measurement what *kind* of motion is produced by force in mass; we can thus test the statement that force produces in mass that kind of motion which has been defined as uniformly increasing velocity.

II. We can test by observation the statement that mass when acted on by no resultant external force moves with uniform velocity.

III. Since the mass of the string is very small, it causes no appreciable effect on the tension; if we also neglect the effects produced by the mass of the wheel and the friction of the axle (which are both comparatively small), then the tension remains unchanged throughout the whole length of the string. The string may then be described as *the means of communicating* the two portions of a stress which act one on each of the masses; in this way, by comparing the actual motion of the masses with the motions given by our calculations, we have a practical test of the truth of the Third Law of Motion.

Example i. *In an* Atwood's machine *the masses suspended* A *and* B, *each contain* M lbs.; *a small mass* C *of* m lbs. *is placed on the mass* A: *find the resulting* acceleration *and the* tension *of the string.*



I. Consider the motion of the masses (A and C), treating them as one mass.†

The external forces acting are

(i) the weight of $(M+m)$ lbs.;

that is, $(M+m)g$ poundals,

(ii) the tension of the string; let this be T poundals.

The weight acts vertically downwards and the tension acts vertically upwards; hence, the resultant external force is

$$\{(M+m)\,g-T\} \text{ poundals } \textit{downwards}.$$

Let (A and C) have α celos downwards; then it has $(M+m)\,\alpha$ pound-celos; so that the force acting upon it must be

$$(M+m)\,\alpha \text{ poundals } \textit{downwards}.$$

Therefore, $(M+m)g-T=(M+m)\,\alpha$ (i).

II. Consider the motion of the mass B.

The string being inextensible, whatever motion A may have, B must have an equal motion in the opposite direction; therefore B has an acceleration $-\alpha$ celos downwards.

The mass acceleration of B is therefore

$$-M\alpha \text{ pound-celos, } \textit{downwards}.$$

So that it is acted on by $-M\alpha$ poundals.

The resultant external force is $(Mg-T)$ poundals, *downwards.*

For the tension of the string is a stress which acts equally on A and B in opposite directions.

Therefore $$Mg-T=-M\alpha \qquad \text{(ii)}.$$

Hence, by subtraction of (ii) from (i),

$$mg=(2M+m)\,\alpha,$$

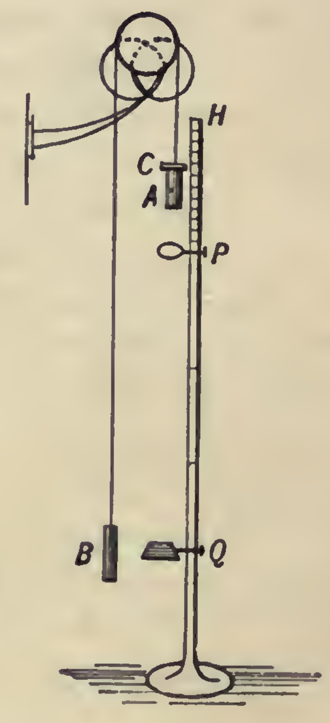
that is, $$\alpha=\frac{m}{2M+m}g \qquad \text{(iii)}.$$

Also from (ii) $$T=Mg+M\alpha, =\frac{2M(M+m)}{2M+m}g.$$

† See note on p. 45.

Example ii. *In an* Atwood's machine *the masses* A *and* B *are each* $15\frac{1}{2}$ oz. *and a mass* C *of* 1 oz. *is placed gently on the mass* A *when at rest; after* 2 seconds *C is removed; discuss the resulting motion.*

I. Consider the motion of the masses *A* and *C*, considering them as one mass.



The external forces are (the weight), $\frac{33}{32} \times 32$ poundals, and (the tension) $-T$ poundals *downwards;* let a celos *downwards* be the acceleration, the mass is $\frac{33}{32}$ lbs.; the mass-acceleration is $\frac{33}{32} \times a$ pound-celos; whence, as in Ex. i,

$$33 - T = \tfrac{33}{32} a \qquad \text{(i)}.$$

II. Consider the motion of the mass *B*; then the forces are $\frac{31}{32} \times 32$ poundals, and $-T$ poundals, *downwards;* the mass is $\frac{31}{32}$ lbs., the acceleration is $-a$ celos *downwards;* therefore, as in Example i, we have

$$31 - T = -\tfrac{31}{32} a \qquad \text{(ii)}.$$

Whence by subtraction

$$2 = \tfrac{64}{32} a = 2a,$$

or

$$a = 1 \qquad \text{(iii)}.$$

Thus, as long as the mass *C* forms part of the mass *A* each mass has 1 celo, the mass *B* upwards and the masses *A* and *C* downwards.

Hence at the end of 2 seconds each mass will have 2 velos and will have moved over ($\frac{1}{2} at^2$), that is, 2 feet.

Forming the equations (i) and (ii) on the supposition that *C* is not put upon *A*, we have

$$\left.\begin{aligned} 31 - T &= \tfrac{31}{32} a \quad [T \text{ here is not the same} && \text{(i)}\\ 31 - T &= -\tfrac{31}{32} a \quad \text{as the } T \text{ above.]} && \text{(ii)} \end{aligned}\right\}$$

and,

whence

$$a = 0.$$

So that when, at the end of 2 seconds, the 1 oz. mass is removed, the acceleration vanishes; that is, the velocity is uniform; and the masses each move with 2 velos, *A* downwards and *B* upwards.

Example iii. *Find the* stress *between the small* mass C *and the* mass A *in the above Example* (ii).

The mass *A* and the mass *C* each descends with 1 celo.

The weight of the mass *C*, viz. (of $\frac{1}{16}$ lb.) 2 poundals acting on $\frac{1}{16}$ lb. would produce in it 32 celos. It actually has 1 celo.

Hence the stress is such that the upward part of it acting on $\frac{1}{16}$ lb. destroys in it 31 celos.

The upward action of the stress is therefore $\frac{31}{16}$ poundals.

Example iv. *A horizontal platform is made to descend vertically with* 4 celos; *a man stands on the platform with* 56 lbs. *in his hand; shew that this mass appears to weigh only* 49 pounds.

The force acting on the 56 lbs. (besides the pressure of the hand) is its weight downwards, that is,

$$56 \times 32 \text{ poundals.}$$

This would produce in it 32 celos, if the upward force of the man's hand did not prevent. The 56 lbs. actually has 4 celos, which requires

$$56 \times 4 \text{ poundals to produce it.}$$

Thus, of the weight (56×32 poundals) the part 56×4 poundals is taken up in producing the 4 celos; so that the upward force of the man's hand must counteract the remainder. In other words this force is

56×28 poundals; that is, the weight of 49 lbs.

57. To find the acceleration in an Atwood's machine we may proceed as follows.

Let the masses be m lbs. and m' lbs.

Then, considering the string as the means of preventing relative motion between the masses, and therefore the mass moved to be the sum of the masses m lbs. and m' lbs., the tension of the thread becomes an internal stress and has no effect on the mass-acceleration produced.

The resultant external force in the direction of the motion of the particles is (the weight of the greater mass m − that of the less m'), hence we have $$mg - m'g = (m + m')\,a.$$

58. The problem of Atwood's machine is similar to the following.

Two particles A and B of masses m lbs. and m' lbs., connected by a light inextensible string, are placed with the string stretched, on a smooth horizontal plane; A is acted on by a horizontal force mg poundals in the direction BA, and B is acted on by a horizontal force $m'g$ poundals in the direction AB; find the tension of the string and the acceleration of the masses. It will be found [Art. 55] that the tension is

$\frac{2mm'}{m+m'}g$ poundals; and that the acceleration is $\frac{m-m'}{m+m'}g$ celos.

EXAMPLES. XVII.

Find the tension of the string and the acceleration of the masses in an Atwood's machine, when the masses A and B have the following values:

1. Mass of A, 1 lb. 1 oz.; of B, 15 oz.

2. Mass of A, 5 lbs.; of B, 3 lbs.

3. Mass of A, 25 oz.; of B, 23 oz.

4. Mass of A, 4 lbs. 4 oz.; of B, 3 lbs. 12 oz.

5. In an Atwood's machine the masses A and B are 3 lbs. 15 oz. each; a mass of 2 oz. is gently placed on A and is removed by a catch after A has descended 1 ft.; shew that A will take 4 seconds to descend the next 4 ft.

6. In an Atwood's machine the masses are each 7 lbs. 15 oz.; a mass of 2 oz. is gently placed on one of them and is removed by a catch after descending 8 ft.; how long will it take to pass over the next foot?

7. In an Atwood's machine it is observed that after descending 8 ft. from rest the masses have a velocity of 4 ft. per second; the larger mass is $32\frac{1}{2}$ oz.; find the other mass.

8. In an Atwood's machine the masses are equal and it is observed that a weight of 1 oz. added to one of them produces 1 celo; what are the masses?

9. In an Atwood's machine the smaller mass is 10 lbs.; what is the larger when the acceleration is 3 celos?

10. In an Atwood's machine the acceleration is 2 celos; what is the ratio of the weight?

11. Prove that in an Atwood's machine when the tension of the string is one-fourth of the sum of the weights $m : m' = 3 + 2\sqrt{2} : 1$.

12. Two masses m lbs. and m' lbs. connected by an inextensible thread, are placed with the thread straight on a smooth horizontal plane; the mass m is acted on in the direction of the string away from m' by a force f poundals, and m' is acted on by a force f' poundals in the opposite direction; find the acceleration of the masses.

13. Prove that if in Ex. **12** the forces are equal to the weights of the masses on which they act, the acceleration is $\frac{m-m'}{m+m'}g$ celos.

14. A bullet fired directly into a block of wood will penetrate 3 in.; what part of its velocity will it lose in passing through a board of the same wood 1 in. thick supposing the resistance uniform?

15. A mass of 1 cwt. is placed on a passenger lift which descends vertically with a uniform acceleration of 10 celos; find the pressure between the lift and the mass.

16. If the lift in Question **15** is ascending vertically with an acceleration 16 celos, what is then the pressure?

17. A man can raise a sack of corn weighing $1\frac{1}{2}$ cwt.; shew that when on a lift which is made to descend vertically with 8 celos, he can raise 2 cwt.

18. A balloon ascends vertically with a uniformly accelerated motion so that a weight of 1 lb. produces, on the hand of the aeronaut sustaining it, a downward pressure equal to that which 17 oz. produces when at rest; find the height which the balloon attains in 1 min. from rest.

19. Find the pressure of *A* on the mass of 2 oz. in Question **5** during the motion.

20. Find the magnitude of the stress between the 2 oz. and the mass on which it is placed in Question **6**.

21. A man in a lift holds in his hand 2 lbs. which presses his hand with a vertical force of 2 lbs. 2 oz.; what do we know of the motion of the lift?

22. A lift of 1 ton is descending a shaft which is 100 feet deep; the chain by which it is suspended has a uniform tension equal to the weight of 15 cwt.; if the lift started from rest at the top of the shaft, with what velocity would it reach the bottom?

23. Suppose the lift in Question **22** is counterbalanced by a weight at the other end of the chain, the chain passing over a fixed pulley; what must be the weight that the tension may be equivalent to the weight of 15 cwt.?

24. A mass of 10 lbs. is fastened to one end of a light thread which passes over a smooth pulley; what weight must be attached to the other end in order that the 10 lbs. may ascend 8 ft. from rest in 4 seconds?

25. A man in a lift at rest holds in his hand 1 lb.; suddenly the 1 lb. appears to weigh 15 oz.; then suddenly the weight appears to change to that of 17 oz.; and next it appears to have its usual weight; he finds the lift has descended 128 ft. from rest and has come to rest. How long did it take in the descent?

26. In an Atwood's machine the pulley is suspended by a hook from a nail; shew that (with the usual notation) the force on the nail is $\left(\frac{4mm'g}{m+m'}\right.$ poundals + the weight of the pulley$\left.\right)$.

27. Shew that in an Atwood's machine the tension of the string diminishes as the ratio of the larger mass to the smaller increases, their sum remaining unaltered.

CHAPTER IV.

IMPULSE. IMPACT. ELASTICITY.

59. By Art. 36, *Mass* never changes its uniform velocity in a straight line, except it be acted on by *continuous force*.

By Art. 31, *Force* produces in mass on which it acts *acceleration*.

Therefore the velocity of Mass **cannot** be changed *instantaneously*.

For by Art. 18 acceleration signifies a *growth* of velocity, and an acceleration *always* requires an interval of time in which to produce a finite change of velocity.

Hence, *Mass* [see Art. 49] is such that in order to produce a finite change in its velocity, two conditions are necessary; **a continuous force** must act upon it, and **an interval of time** must elapse.

60. DEF. An **impulse** is that which produces in the mass on which it acts a finite velocity, so that the mass-velocity produced varies as the impulse producing it.

61. DEF. We choose as our **unit** *impulse* that impulse which acting on a pound produces in it 1 velo.

We shall call the *unit impulse* a **pulse**.

[Compare Arts. 40, 134.] Hence the number of *pulses* in an impulse is equal to the number of *pound-velos* which it produces, so that Art. 60 may be stated thus.

DEF. An *impulse* is numerically equal to and is in the same direction as the *mass-velocity* which it produces.

62. Since an impulse produces a finite change in the velocity of mass, *an impulse* is by Art. 59 the joint effect of a continuous force and of an interval of time.

In what follows we shall only consider those impulses which are produced by *constant* force.

A pulse has the same effect on a mass as a poundal acting continuously throughout 1 second; each produces a pound-velo in the mass on which it has acted. It has the same effect as (i) 2 poundals acting for $\frac{1}{2}$ second;
as (ii) 100 poundals acting for $\frac{1}{100}$th of a second;
as (iii) 10000 poundals acting for $\frac{1}{10000}$th of a second;
and so on.

A *pulse* might be termed a *poundal-second;* meaning that a pulse has the same effect as a poundal acting continuously for a second.

A *pulse* = a *pound-velo* = a *poundal-second.*

The force of an impulse is the force (here supposed constant) whose continued action for some interval produces the impulse.

The interval of an impulse is the interval throughout which the action of the force is continued to make the impulse.

Example i. *A* cricket ball *of* $\frac{1}{2}$ lb. *receives an additional velocity of* 100 velos; *what* impulse *has been applied to it?*

Its mass velocity is increased by $\frac{1}{2} \times 100$ pound-velos,

∴ 50 pulses have been applied to it.

Example ii. *If in* Example i. *the* interval *of the impulse were* (i) $\frac{1}{2}$ of a second, (ii) $\frac{1}{10}$th of a second, (iii) $\frac{1}{1000}$ of a second, *find the* force of the impulse (*supposed uniform*) *in each case.*

(i) The acceleration necessary to produce 100 velos in $\frac{1}{2}$ second is 200 celos. Hence the acceleration of the cricket ball in this case was 200 celos. The mass is $\frac{1}{2}$ lb.,

∴ the force is $\frac{1}{2} \times 200$ poundals; or 100 poundals = $3\frac{1}{8}$ pounds weight.

(ii) The acceleration necessary to produce 100 velos in $\frac{1}{10}$ sec. is 1000 celos,

∴ the force is 500 poundals; or $15\frac{5}{8}$ pounds weight.

(iii) The acceleration necessary to produce 100 velos in $\frac{1}{1000}$ sec. is 100000 celos,

∴ the force of the impulse is 50000 poundals; or $1562\frac{1}{2}$ pounds weight.

EXAMPLES. XVIII. a.

1. A train of 100 tons is observed to be moving on smooth horizontal rails with a velocity of 15 miles an hour; after an interval it is observed to be moving with a velocity of 45 miles an hour; what impulse has been applied to it?

2. If in Question **1** the interval is 5 minutes and a uniform force is acting on the train, what is the force?

3. If in Question **1** a uniform horizontal force (beyond what is necessary to overcome friction, etc.) of 28 lbs. weight is acting on the train, what is the interval?

4. An ironclad war ship of 10000 tons is moving with velocity of 18 miles an hour, and after an interval it is at rest, what impulse has acted upon it? also supposing the interval to have been 5 minutes, what average force (beyond that necessary to overcome friction etc.) has been acting upon it?

5. If in Question **4** the interval were 10 seconds; what is the force?

6. An ironclad of 8000 tons attains from rest a velocity of 15 miles an hour; what is the impulse necessary? If the interval required is 5 minutes, what is the average resultant force (beyond that necessary to overcome friction etc.) which its engines can obtain from the blades of the screws?

7. A railway engine of 50 tons impinges on a sandbank with a velocity of 60 miles an hour; what impulse does it apply to the sandbank? If it is brought to rest in $\frac{1}{2}$ a second, what pressure does it apply to the sandbank?

8. A football of $\frac{1}{2}$ lb. is moving just after it has been kicked from rest with 20 velos; what impulse has it received? and find the force of the impulse supposing the interval of the impulse to have been (i) $\frac{1}{2}$ sec., (ii) $\frac{1}{10}$ sec., (iii) $\frac{1}{1000}$ sec.

9. A sculler can apply to the water with the blades of his sculls a pressure of 28 lbs. weight; supposing that he rows 30 strokes a minute and that the pressure is applied to the water for a quarter of each stroke, find what impulse he applies in 1 minute.

10. If in **9** the mass of the sculler and his boat is 4 cwt.; supposing the force necessary to overcome the friction of the water to be 21 lbs. weight, in how many strokes will the sculler attain a velocity of 15 miles an hour?

11. A tricyclist and his machine are 2 cwt.; find the pressure on the pedals (neglecting friction, the rotary motion of the wheels, etc.) supposing that he attains a velocity of 15 miles an hour from rest in 20 seconds, and that he applies the pressure for two-thirds of each revolution of the wheels.

63. When a mass has received a finite addition to its velocity we may say either (i) that it has received an impulse; or (which is the same thing) (ii) that it has received the continuous action of a force for a finite interval.

That which in ordinary language we are accustomed to call a **blow** of a bat, or a **stroke** of an oar, or a **tread** of a tricyclist's foot on a pedal, each is an impulse. They each consist of a stress lasting for an interval.

One component force of the stress acting on the ball, or on the blade of the oar, or on the pedal of the tricycle, for an *interval* produces a certain number of pound-velos, in the ball, or in the boat, or in the tricycle.

It is well known both to cyclists and to oarsmen that the *longer the interval* of the impulse (the force being the same) the greater the mass-velocity produced.

Hence the importance of the *length* of the stroke in rowing.

Example. A shot of 8 cwt. *from a* 38 ton gun *is observed to leave the gun with a velocity of* 2000 velos; (i) *find the* impulse *it has received;* (ii) *if the shot passes over* 10 ft. *when in the tube of the gun, find the resultant force which acted on the base of the shot, supposing that force to have been* uniform.

The shot has received $2000 \times 8 \times 112$ pound-velos.

Therefore $\underline{2000 \times 8 \times 112 \text{ pulses}}$ have acted on it. (i)

The force whose continuous action produced this mass-velocity is supposed to have been uniform; so also therefore must have been the acceleration; also the shot passed over 10 ft. while in the gun.

Let the acceleration be a celos,

then, [Art. 26, iii] $\quad a \times 10 = \frac{1}{2}v^2 = \frac{1}{2} \times (2000)^2,$

whence, $\quad a = 200000.$

The mass of the shot is 8×112 lbs.; so that its mass acceleration is

$8 \times 112 \times 200000$ pound celos.

To produce this, $\quad 8 \times 112 \times 200000$ poundals are necessary.

The force is therefore $\quad 179200000$ poundals,

equivalent to <u>the weight of</u> $\dfrac{8 \times 112 \times 200000}{32 \times 20 \times 112}$ tons;

that is, of $\underline{2500 \text{ tons}}$. (ii)

64. Suppose the interval occupied by q pulses is t seconds; q pulses produce q pound velos. Also, $\frac{q}{t}$ *poundals* continued throughout t seconds produce q pound velos; therefore, the average *force* whose continued action produces q pulses in t seconds is $\frac{q}{t}$ poundals.

Hence, the measure of the average force of a given impulse is the ratio of the number of pulses acting to the number of seconds in the interval.

In other words *the average force of an impulse is measured by the* **rate** *of impulse per second.*

Example. A glass ball of 4 oz. *on a smooth horizontal pavement strikes a vertical wall at right angles with* 50 velos, *and is observed to rebound with* 30 velos; *find the* impulse *which it receives; supposing the* impulse *occupies* $\frac{1}{100}$th of a second, *find the* average magnitude *of the* stress *between the ball and the wall.*

[NOTE. Here, the velocity on reaching the wall is 50 velos; when that velocity ceases, the ball has received 50 velos in the direction opposite to its original velocity;—for, $+50$ velos and -50 velos produce rest;—when the ball is again moving with 30 velos, it has received 30 velos more; so that in all it has received 80 velos. These velos were the result of an acceleration produced by a force, which acted for some interval; the interval was very small (too small perhaps to be observed) but still an interval. The interval given in the question, viz. $\frac{1}{100}$th of a second, is too small to be actually observed.]

The change in the velocity of the ball is 80 velos; therefore the mass-velocity produced is $\frac{4}{16} \times 80$ pound-velos; that is, 20 pound-velos, and this requires an impulse of 20 pulses.

Let x be the *average* number of poundals in the stress; then,
x poundals acting on $\frac{1}{4}$ lb. produce $4x$ celos,
$4x$ celos in $\frac{1}{100}$ secs. produce $\frac{4x}{100}$ velos;

hence, $$\frac{4x}{100} = 80; \text{ or, } x = 2000.$$

The *average* force on the ball is 2000 poundals;
that is, the weight of $\frac{2000}{32}$ pounds; that is, of $62\frac{1}{2}$ lbs.

EXAMPLES. XVIII. b.

1. A shot of 8 cwt. leaves a fixed 40 ton gun with a muzzle velocity of 2000 ft. per second; find the impulse which has acted upon it. Also if the shot moves over 20 ft. when in the gun, find the resultant uniform push which has acted on the base of the shot.

2. Find how long the shot in Question 1 was in motion in the tube of the gun.

3. On Friday Feb. 11th, 1886, at Woolwich Arsenal a gun of 111 tons was fired for the first time with an iron shot of 1800 lbs.; the muzzle velocity was 1600 velos; find the impulse which acted on the shot. Also if the shot moved over 40 ft. while in the gun, find the resultant uniform push which was applied to its base to produce this velocity.

4. Find how long the shot in Question 3 was in motion in the tube of the gun.

5. A glass ball of 1 lb. on a smooth horizontal pavement strikes a vertical wall at right angles with 20 velos and leaves it with 12 velos; what impulse has acted on the ball? and supposing the ball and wall were in contact for $\frac{1}{10}$ sec., find the average external force acting on the ball.

6. An india-rubber ball weighing 1 lb. on a smooth horizontal pavement strikes a vertical wall at right angles with a velocity of 30 velos, and rebounds with a velocity of 25 velos; find the impulse which has acted upon it; and find what force acting for $\frac{1}{10}$th of a second would have the same effect.

7. Two equal railway carriages of 3 tons moving in opposite directions each with a velocity of 10 miles per hour meet, and rebound each with a velocity of 5 miles an hour; find the impulse which has acted on each. Find also in tons weight the force between the buffers, supposed uniform, assuming that they were in contact for a second.

8. A hammer of 2 lbs. hits a nail with a horizontal velocity of 40 ft. per second and drives the nail into a deal board to the depth of half an inch; find (i) the force which is exerted on the nail supposing it uniform, (ii) the duration of the impulse.

9. A hammer of 14 lbs. strikes a nail with a *vertical* velocity of 40 velos and comes to rest in $\frac{1}{16}$th of a second; find how far it moves the head of the nail, and the pressure on the nail, supposed uniform.

10. A steam hammer of 10 tons is let fall from the height of 4 ft., and strikes a mass of red-hot iron; supposing that the hammer on touching the iron comes to rest in $\frac{1}{32}$nd of a second, and that the force it exerts is uniform, find the push, and by how much it flattens the red-hot iron.

11. An iron weight of half a ton is let fall from a height of 16 ft. vertically above the head of a pile; the pile is driven into the ground a distance of 1 ft. by the impulse; find (i) the average force, (ii) the duration of the impulse.

*IMPACT.

65. When one mass meets another mass having a different velocity the masses are said to **impinge**.

The meeting of the two masses is called an **impact**.

The impact is said to be **direct** when the masses are moving in the same straight line, provided the direction of the stress caused by the impact is also in that straight line.

Let us consider the case of one mass having a certain velocity impinging on another mass having a different velocity, the masses being each composed of a hard substance, such as a metal or glass.

In one or both of the masses, since the substances can yield only very slightly at the points in contact, there must be a very rapid change of velocity; that is, there must be a very great acceleration; consequently there must be an immense *mutual pressure* between the masses.

In every such case the impact must, by Art. 59, occupy some definite interval of time.

The particles of each mass yield slightly in the neighbourhood of the great pressure, and then, either return more or less exactly to their original position (in which case the material is said to be **elastic**), or do not return (in which case the impact causes a permanent indentation, and the material is said to be **inelastic**).

66. An impact between two masses causes a stress to be set up between them; which stress lasts for a small interval. The action and reaction of this stress produce *equal* external impulses which act on the two masses in opposite directions.

These equal and opposite impulses produce equal and opposite mass-velocities, one in one mass, the other in the other. Hence, the algebraic sum of the *additional* mass velocities produced during an impact is zero; in other words,

when two masses impinge their total mass-velocity is unaltered by the impact. [See p. 50.]

67. DEF. Two masses are said to be **inelastic** when, after direct impact, their velocities after impact are equal in every respect.

NOTE. When one mass **sticks** to another on impact; or when one mass **penetrates** another, as a bullet penetrates a piece of wood, and stays in it; in fact in every case in which the masses after impact *move as one mass*, the masses are said to be *inelastic*.

Example i. *A mass* M *of* 3 lbs. *having* 5 velos *impinges directly on a mass* M′ *of* 7 lbs. *at rest; find their subsequent motion, the masses being* inelastic.

Since the masses are inelastic, their velocities are made equal by the impact; let their velocity after impact be v velos.

Before impact $\begin{cases} \text{the mass-velocity of } M \text{ is } 3 \times 5 \text{ pound-velos.} \\ \text{the mass-velocity of } M' \text{ is } 0. \end{cases}$

After impact $\begin{cases} \text{the mass-velocity of } M \text{ is } 3 \times v \text{ pound-velos.} \\ \text{the mass-velocity of } M' \text{ is } 7 \times v \text{ pound-velos.} \end{cases}$

Hence, by Art. 67, $3v + 7v = 15$;

or, $v = \frac{3}{2}$; that is,

after impact they move together with a velocity of $1\frac{1}{2}$ ft. per second.

Example ii. *A leaden ball of* 1 lb. *impinges directly with a velocity of* 1000 ft. per sec. *on a mass of* 1 cwt. *moving in the same direction with a velocity of* 20 ft. per sec.; *find their subsequent velocity.*

Let them have v velos after the impact; then, since the total mass-velocity is unchanged

$$1 \times 1000 + 112 \times 20 = v \times (112 + 1),$$

or, $v = \frac{3240}{113} = 28{\cdot}6\ldots$ That is,

the masses move together with a velocity of 28·6... ft. per second.

68. It should be noticed [Art. 62] that the smaller the *interval* of a given impulse, the larger is the average *force* which produces that impulse.

Also, the smaller the *interval* (other circumstances being equal), the smaller is the *distance* passed over by the mass during the impulse.

For, let m lbs. moving with u velos have mv pulses applied to it in t seconds; then, assuming that the v velos are produced uniformly, $s = (u + \frac{1}{2}v) \times t$; so that the smaller t, the smaller is s.

Example i. *An engine of* 60 tons *impinges with the velocity* $7\frac{1}{2}$ miles per hour ($=11$ velos) *on a carriage of* 10 tons *at rest. The breaks of the engine are exerting a horizontal retarding force of* k tons weight (*the rest of the* friction, *the* rotary motion *of the wheels* etc. *is neglected*). *At the instant at which the carriage attains an equal velocity with the engine it is firmly coupled up with it. Assuming that the pressure between the buffers during impact is constant and that the interval occupied by the impact is the* nth part of a second, *find the* pressure; *find also the* velocity of the carriage at the instant of the coupling.

By Art. 66 the pressure of the engine on the carriage is equal and opposite to the pressure of the carriage on the engine.

The carriage has a positive acceleration in the direction of the velocity of the engine; the engine has a negative acceleration in the same direction. Let v velos be their joint velocity after impact.

The acceleration of the carriage is nv celos [for $v=at$].

Therefore the pressure is $[ma]=10\times 2240\times nv$ poundals.

The engine loses $(11-v)$ velos in the nth part of a second,

$\therefore$ its acceleration is $\quad -(11-v)\,n$ celos,

$\therefore$ it has $\quad -(11-v)\,n\times 60\times 2240$ pound-celos.

This acceleration is produced partly by the pressure of the carriage and partly by the retarding force of the breaks,

that is, by $\quad (10\times 2240\times nv+2240\times 32k)$ poundals,

which force produces $-(10\times 2240\times nv+2240\times 32k)$ pound-celos,

$$\therefore\ 10\times 2240\times nv+2240\times 32k=(11-v)\times 60\times 2240n,$$

whence,
$$v=\frac{660n-32k}{70n}=\frac{66}{7}-\frac{32k}{70n},$$

$$\therefore \text{ the required pressure is } 10\times 2240\times\frac{660n-32k}{70}\text{ poundals}$$
$$=320\times(660n-32k)\text{ poundals.}$$

NOTE. In such an experiment k would rarely exceed 3, while the interval $\left(\frac{1}{n}\text{ sec.}\right)$ is small, especially if we suppose the buffers of the engine and carriage to be without springs. In this case n may be a large number. When n is large and k small we see that the effect of the retarding force of the breaks on the pressure and on the ultimate velocity is *comparatively* small, while the velocity after impact does not differ much from $\frac{66}{7}$ velos, that is $9\frac{3}{7}$ velos.

In the above example, taking for the interval $\frac{1}{100}$ of a second and for the retardation of the breaks 3 tons weight,

the actual pressure is $320 \times (6600 - 64)$ poundals,

the actual velocity after impact $\dfrac{6600-64}{700} = \dfrac{6536}{700} = \underline{9\tfrac{236}{700}\text{ velos.}}$

Consider the following *Example* ii. *An engine of* 60 tons *moving with the velocity* 11 velos, *impinges on a carriage of* 10 tons *at rest; find the* velocity after impact *supposing the engine and carriage to be* inelastic.

We observe that the length of the interval of the impact is not given; that the *pressure* is not asked for; and we are not told what external forces, if any, are acting upon the engine.

In such a question it is supposed that the interval of the impact is so small, and consequently that the pressure between the masses is so large, that (in comparison) the effect, for the interval of the impact, of the ordinary forces of friction etc. may be neglected. Thus

If v be their joint velocity after impact, we have, since action and reaction are equal and opposite, $v(m+m') = 0 \times m + 11 \times m'$,

$$\therefore\ v(70 \times 2240) = 11 \times 60 \times 2240,$$

or $v = \frac{660}{70} = 9\frac{3}{7}$. $\therefore$ their joint velocity after impact is $\underline{9\tfrac{3}{7}\text{ velos.}}$

This is true, no matter what may be the length of the interval occupied, when *no external horizontal forces* act upon the masses.

This is nearly true, provided the interval is *small*, *whatever* ordinary external forces act upon the masses; and it becomes more and more nearly true the smaller is the interval occupied by the impulse; hence,

69. When a question on impact does not contain sufficient data to determine the magnitude of the interval and the average pressure during the impact, it is understood that *the interval is so small* that the effect of all ordinary external forces for this interval, is so much smaller than that of the pressure of the impact, that the effect of these external forces may be neglected.

This is all that is meant when it is said that '*the force of an impact is infinite compared with ordinary forces.*'

This supposition is made in Ex. XIX. 12 in Art. 73; in Ex. XXI.; etc.

EXAMPLES. XIX.

The following impacts are all direct.

1. An inelastic ball of 1 cwt. moving with a velocity of 20 ft. per sec. impinges against an equal ball at rest; find their subsequent velocity.

2. An inelastic particle of m lbs. having v velos impinges directly on another of m' lbs. at rest; find their subsequent velocity.

3. An inelastic particle of 20 lbs. meets another of 2 lbs.; each particle has 11 velos; find their subsequent motion.

4. A railway carriage of 10 tons is moving towards a train of 100 tons with a velocity of 4 miles an hour; at the instant of collision the carriage is coupled to the train; find the subsequent velocity.

5. A particle of 1 cwt. meets another of 28 lbs. having a velocity of 15 miles an hour and after the impact they both are at rest; with what velocity was the 1 cwt. moving?

6. A particle of 1 lb. moving with 30 velos overtakes another particle of 2 lbs. and after impact they move together with 25 velos; what was the velocity of the 2 lbs. before impact?

7. An inelastic mass impinges on another of twice its mass at rest; shew that the impinging body loses two-thirds of its velocity by the impact.

8. An inelastic particle of mass m and velocity v impinges directly on a particle of mass m'; they are at rest after the collision; shew that the velocity of the second particle was $-\frac{m}{m'}v$.

9. Three equal inelastic balls are placed in a line, not in contact; an equal ball moving in the same line with 20 velos impinges on the first, these two impinge on the second and then the three on the third; find their joint velocity after impact.

10. An engine of m tons impinges with a velocity of 4 miles an hour on each of 4 inelastic trucks each of $\frac{1}{20}$ m tons placed on the same line of rails and separated by a small interval; find the velocity of the train after impact.

11. A rifle bullet of $\frac{1}{2}$ oz. having 1200 velos impinges on a block of wood weighing 5 cwt. at rest and free to move in the direction of the impact; what is the velocity of the block immediately after impact?

12. A particle is let fall from the top of a tower and at the same instant an equal particle is thrown vertically upwards from the foot of the tower with a just sufficient velocity to carry the particle under the action of gravity to the top of the tower. The two particles being inelastic impinge; shew that directly after impact the particles are at rest; and if the tower is 128 ft. high find how long the first particle takes to reach the ground.

13. Shew that the engine and carriage in the Example on p. 66 will come to rest after an interval which is *independent of the duration of the impact.*

* ELASTICITY.

70. DEF. Two masses are said to be **elastic** when, after impact, their velocities in the direction of the impulse are not the same.

When two *elastic* masses impinge they probably behave in much the same way as inelastic masses up to the instant at which the two masses are moving with the same velocity; the property of elasticity seems to cause a continuation of the impulse beyond that instant so that the masses *recoil* from each other. The amount of recoil must in some way depend on the magnitude of the impulse and on the nature of the material of the masses.

71. Newton observed, that when two spheres of given substance impinge directly on each other, their *relative* velocity after impact is e times their relative velocity before impact and in the opposite direction; where e is a number not greater than 1. This number e depends on the *nature* of the substances of the two spheres, and is called their **coefficient of restitution**; hence we say that

72. When two elastic masses m lbs. and m' lbs. impinge directly on each other, u velos and u' velos being their velocities before impact, v velos and v' velos their velocities after impact, then

$$v - v' = -e(u - u');$$

where e is a number, not greater than 1, which depends on the materials of the masses and is called the *coefficient of restitution.*

In other words, the two masses separate after impact with a velocity which is e times that velocity with which they were approaching each other before impact.

Velocity of separation $= e$ times velocity of approach.

When the coefficient of restitution of a substance $= 1$ the substance is said to be **perfectly elastic.**

Example i. *Two masses of* 5 lbs. *and* 20 lbs. *having velocities of* 20 velos *and* -10 velos *respectively, and whose* coefficient of restitution *is* $\frac{2}{3}$, *impinge; find their subsequent velocities.*

Let v velos and v' velos be their velocities respectively after impact.

Then, since their total mass-velocity is unaltered by the impact

$$5\times 20-20\times 10=5v+20v' \qquad \text{(i)}$$

also, by Art. 41,
$$v-v'=-\tfrac{2}{3}(20+10) \qquad \text{(ii)}$$

whence
$$5v+20v'=-100,$$

and
$$v-v'=-20;$$

$$\therefore\ 25v=-100-400;\ \text{or, } v=-20;\ \text{also, } v'=0.$$

Hence, after the impact the 5 lbs. moves in the negative direction with velocity 20 ft. per second, and the 20 lbs. is at rest.

Example ii. *Two masses of* m lbs. *and* m′ lbs. *impinge with velocities* u velos *and* u′ velos; *find the condition that they may interchange velocities.*

Here u' velos and u velos are to be their respective velocities after impact;

hence,
$$mu+m'u'=mu'+m'u \qquad \text{(i)}$$

or,
$$(m-m')(u-u')=0,$$

whence,
$$m=m'$$

(for, since the masses impinge, u cannot $=u'$).

Again,
$$u-u'=-e(u'-u) \qquad \text{(ii)}$$

or,
$$(u-u')(1-e)=0,$$

whence,
$$e=1.$$

Thus, in order that two masses after impact may interchange velocities they must be of equal mass and perfectly elastic.

EXAMPLES. XX.

The coefficient of restitution is here denoted by e.

1. A sphere of 6 lbs. having 20 velos overtakes another of 4 lbs. having 12 velos; determine their velocities after impact, in the case in which $e=\frac{1}{2}$.

2. A particle of 28 lbs. having 8 velos impinges directly on another of 14 lbs. having -16 velos; determine their subsequent velocities when $e=\frac{1}{3}$.

3. Two elastic balls ($e=\frac{3}{4}$) of masses $3m$ and m and velocities $3v$ and $-5v$ impinge; find their subsequent velocities.

4. A ball of $3m$ impinges on another m at rest which afterwards moves with 10 velos, e being $\frac{2}{3}$; find the velocity of the first ball before impact.

5. The result of a direct impact between two spheres of elasticity $\frac{1}{4}$, one of which is at rest, is that one of them has after impact twice the velocity of the other; prove that one ball has $\frac{3}{2}$ the mass of the other.

6. A series of equal elastic balls of elasticity e are placed in a line separated from each other by short distances; another ball of equal mass moving in the same line impinges on the first with velocity v; find the velocities after impact of the first, second, third and nth balls.

7. A ball of mass m and elasticity e is projected vertically under the action of gravity with 64 velos; another ball of mass m' is simultaneously let fall from a height of 64 ft. vertically above m; find when and where they impinge, and their velocities after the impact.

8. Two elastic spheres impinge directly with equal velocities; find the ratio of their masses that one of them may be reduced to rest by the impact.

9. A shot of 1 cwt. is projected from a 20 ton gun placed on a smooth horizontal plane, with a horizontal muzzle velocity of 3000 velos; find the velocity of the recoil of the gun.

10. Two railway carriages B and C of m lbs. and m' lbs. stand on the same line of perfectly smooth rails separated by a short distance; a third carriage A of m lbs. impinges on B and then consequently B impinges on C; prove that A will impinge a second time on B if m' is greater than $\dfrac{2em}{1+e^2}$, where e is the coefficient of restitution.

11. Two equal spheres A, B are connected by a string and laid on a smooth horizontal table, at a distance from each other which is less than the length (l ft.) of the string; a velocity u is given to A in the direction BA; on the string becoming tightened a direct impulsive tension is set up; if the coefficient of restitution of the string be e, and of the spheres themselves e', find the velocity of the spheres (i) after the first impulse of the tension, (ii) after the first impact of the spheres.

12. A series of equal masses of m lb. each are connected by light inelastic strings each a ft. long. They are placed in a straight row touching each other on a smooth horizontal plane; to the first mass is applied a force of p poundals throughout the motion in the direction of the row, and when the first mass has moved over a ft., the string applies a direct impulse to the second mass; and when the first two have moved over a ft. more, the string applies a direct impulse to the third mass, and so on; find the initial velocities of the second, third and nth masses.

Example iii. *A small particle of mass* m lbs. *moving with* u velos *impinges directly on a large mass of* M lbs. *at rest; find their subsequent motion, the coefficient of restitution being* e.

We have, with the usual notation,

$$mu = mv + Mv', \qquad \text{(i)},$$

$$v - v' = -eu; \qquad \text{(ii)},$$

whence, $$v = \frac{m - eM}{m + M}u; \text{ and, } v' = \frac{m}{M+m}(1+e)u.$$

Now suppose M to be so much larger than m, that the ratio $\frac{m}{M}$ is too small to be observed, and may be considered to be nothing; then

$$v' = 0; \text{ and,} \qquad v = \frac{\frac{m}{M} - e}{\frac{m}{M} + 1}u = -eu.$$

This is the case when a ball of mass m lbs. comes into collision with the Earth of mass M lbs. Hence we say that

73. When an elastic ball impinges against any surface fixed to the Earth, its velocity after impact is $-e$ times that before impact.

Example. A particle let fall from a height of 16 ft., *impinges on a horizontal pavement, the coefficient of restitution being* e; *it again falls and rebounds; and so on; find when the particle will come to rest.*

The particle has an acceleration 32 celos downwards.

Let its velocity on striking the ground be v velos, and let its velocities after each impact be $-v_1$ velos, $-v_2$ velos etc. respectively; then since a mass thrown vertically upwards passes each point in its descent with the same velocity which it had at that point in its ascent, therefore its velocities on reaching the ground after each rebound are v_1 velos, v_2 velos, etc. respectively; then we have

$$v^2 = 2as = 2 \times 32 \times 16; \quad \text{or, } v = 32.$$

Also, by Art. 73, $v_1 = ev$; $v_2 = ev_1 = e^2v$; $v_3 = ev_2 = e^3v$; and so on.

To ascertain the intervals, we observe that to produce 32 velos, 32 celos require 1 sec.; to produce $e \times 32$ velos, 32 celos require e times 1 sec., and so on.

In the successive bounds ev velos, e^2v velos,... are in turn first destroyed during the upward motion and then reproduced by the acceleration due to gravity in the descent.

Hence the sum of the intervals occupied by the bounds are

$$\{2e + 2e^2 + 2e^3 + \ldots\ldots\ldots\} \text{ seconds.}$$

The limit of this series is $\dfrac{2e}{1-e}$ seconds.

Therefore the particle comes to rest in

$1 + \dfrac{2e}{1-e}$, or, $\dfrac{1+e}{1-e}$ seconds, from the instant at which it was let fall.

EXAMPLES. XXI.

The coefficient of restitution of an elastic ball is sometimes called its elasticity.

1. A particle let fall from a height of 16 feet, impinges on a horizontal pavement; find the interval occupied by the first rebound, the coefficient of restitution being $\frac{1}{2}$.

2. A particle let fall from a height of 48 ft., impinges on a horizontal pavement; find the elasticity that in its first rebound it may rise to a height of 32 ft.

3. A particle let fall from a height of h ft., impinges on a horizontal pavement, the coefficient of restitution being e; find

(i) the height of the first rebound,

(ii) the interval between the first two impacts,

(iii) the time in which the particle comes to rest.

4. An elastic ball falls from a height h ft. to a horizontal plane and then rebounds; falls again and rebounds; and so on; find the sum of the vertical distances passed over.

5. Supposing that in a direct impact between two particles of elasticity e, we consider the impulse between the particles to be divided into two parts, the first part being that which causes the two particles to be relatively at rest; shew from Art. 71 that these two parts of the impulse are in the ratio of 1 to e.

6. The 110 ton gun projects its shot of 1800 lbs. with a muzzle velocity of 2100 velos; what will be the initial velocity of the recoil of the gun, supposing the shot projected horizontally, and the gun to be placed on a smooth horizontal plane?

7. Two masses m lbs. and m' lbs. of elasticity e impinge directly with u velos and u' velos respectively; prove that if v velos and v' velos are their velocities after impact, then

$$v = \frac{(m - em')\,u + m'(1+e)\,u'}{m+m'}$$

and

$$v' = \frac{(m' - em)\,u' + m(1+e)\,u}{m+m'}.$$

*CHAPTER V.

CHANGE OF UNITS.

74. Our units have been, *a foot*, *a second*, *a pound*; from which are derived the units, *a velo*, *a celo*, *a poundal*, *a pulse.*

In all systems of units the velocity chosen for the *unit velocity* is that of a point which passes uniformly over the unit distance in the unit interval.

Similarly, the acceleration chosen for the *unit acceleration* is that of a point whose velocity is increased uniformly by the unit velocity per unit interval.

Also, the force chosen for the *unit force* is that which when acting on the unit mass produces in it unit acceleration.

75. We give some examples involving a change of units.

Example i. *When the* unit acceleration *is* 32 feet per sec. per sec. *and the* unit velocity *is* 10 feet per sec., *find the* unit distance *and the* unit interval.

Let the unit interval be x seconds,

let the unit distance be y feet; then, by Art. 74,

the unit velocity is y ft. per x seconds;

or, the unit velocity is $\frac{y}{x}$ (ft. per sec.); that is, $\frac{y}{x}$ velos. But the unit velocity is 10 ft. per second;

hence, $$10=\frac{y}{x}.$$

The unit acceleration is (unit velocity per unit interval),

that is, 10 velos per x seconds;

or, the unit acceleration is $\frac{10}{x}$ (velos per second); that is, $\frac{10}{x}$ celos. But the unit acceleration is 32 celos;

hence, $$32=\frac{10}{x}; \quad \text{or, } x=\tfrac{5}{16}.$$

Also, $$y=10x=\tfrac{50}{16}=3\tfrac{1}{8}.$$

Thus the unit distance is $3\frac{1}{8}$ ft. the unit interval $\frac{5}{16}$ secs.

Example ii. *When the area of a field of* 10 acres *is the* unit area, *and the* acceleration due to gravity *the* unit acceleration, *find the* unit interval.

A field of 10 acres contains 48400 sq. yds., that is, $(220)^2$ sq. yds. Now the unit area is always the square described on the unit distance;

hence, the unit distance is 220 yds., or 660 ft.

Let the unit interval be x seconds.

By Art. 74, the unit velocity is 660 ft. per x seconds;

that is, $\frac{660}{x}$ velos.

$\therefore$ by Art. 74, the unit acceleration is $\frac{660}{x}$ velos per x seconds;

that is, the unit acceleration is $\frac{660}{x^2}$ celos.

But the unit acceleration is g celos, or 32 celos;

therefore, $$32 = \frac{660}{x^2},$$

or, $$x^2 = \tfrac{165}{8} = 20{\cdot}625,$$

whence, $$x = 4{\cdot}54\ldots$$

The required unit is about $4\frac{1}{2}$ secs.

Example iii. Given, *that the* unit acceleration *is* a celos, *the* unit velocity, u velos; *find the* unit interval *and the* unit distance.

Let the unit interval be x secs.,

let the unit distance be y ft.

Then, by Art. 74, the unit velocity is y ft. per x secs.,

that is, the unit velocity is $\frac{y}{x}$ velos,

$\therefore$ by Art. 74, the unit acceleration is $\frac{y}{x}$ velos per x secs.,

that is, the unit acceleration is $\frac{y}{x^2}$ celos.

But, the unit velocity is u velos, the unit acceleration is a celos.

Hence, $$a = \frac{y}{x^2}, \quad \text{and } u = \frac{y}{x},$$

or, $$x = \frac{u}{a}, \qquad y = \frac{u^2}{a}.$$

Thus, the unit interval is $\frac{u}{a}$ secs., the unit distance is $\frac{u^2}{a}$ ft.

EXAMPLES. XXII.

1. The unit distance being a yard and the unit interval a minute, express the unit velocity in velos and the unit acceleration in celos.

2. The unit distance being a mile and the unit interval an hour, express the unit velocity in velos and the unit acceleration in celos.

3. The unit distance being a yds. and the unit interval b seconds, express the unit velocity in velos and the unit acceleration in celos.

4. The unit velocity being 60 miles an hour, and the unit distance 1 yd., find the unit interval.

5. The unit velocity being 100 yds. per minute and the unit interval 5 secs., find the unit distance.

6. The unit acceleration being the double of that due to gravity and the unit distance 1 mile, find the unit velocity.

7. The unit velocity being 40 ft. in 11 secs. and the unit acceleration 3 yds. per minute per minute, find the units of time and distance.

8. The unit mass being 1 cwt. and the unit force the weight of 7 lbs., the unit velocity 10 velos, find the unit distance.

9. The unit force being the weight of 1 lb., and the unit interval and unit distance being 1 sec. and 1 ft. respectively, find the unit mass.

10. The unit mass being 1 ton. the units of interval and of distance an hour and a mile, find the unit force.

11. Find the measure of the acceleration due to gravity when the unit distance is a metre, say 3·28 ft., and the unit interval a second.

12. What must be the unit interval so that with a foot as the unit distance the acceleration of gravity may be 2?

13. The acceleration of gravity being 12 and the unit interval 5 seconds, find the unit distance.

14. If the unit impulse be the impulse necessary to cause 1 lb. to rise vertically 1 foot under the action of gravity, the unit mass 1 lb., the unit distance 1 foot, what is the unit interval?

15. If the acceleration due to gravity be unity, and the area of a field of 10 acres the unit area, find the unit interval.

16. The unit impulse being that required to bring 10 lbs. to rest after falling 5 feet, the unit distance being 5 yards and the unit interval 3 seconds, find the unit mass.

17. If the unit distance is n feet, the unit interval n seconds, the unit mass n lbs., shew that the acceleration of gravity is ng.

18. If f is the measure of an acceleration when m secs. and n ft. are the unit interval and unit distance respectively, express the acceleration in celos.

19. The unit impulse being that necessary to produce v velos in m lbs., the unit distance being l feet, the unit mass being k lbs., find the unit interval.

20. If f and F be the measures of the same acceleration when the unit interval and unit distance are t secs. s ft. and τ secs. and σ ft., respectively, prove that $$F = \frac{s}{t^2} \times \frac{\tau^2}{\sigma} \times f.$$

21. Taking the centimetre ($\frac{1}{100}$ of a metre $= \cdot 0328\ldots$ft.) as the unit distance, the second as the unit interval and the gram ($= \cdot 0022\ldots$ lbs.) as the unit mass, find the unit velocity in velos, the unit acceleration in celos, and the unit force in poundals.

N.B. This system of units is called the **C.G.S.** [**Centimetre, Gramme, Second**] system; the unit force is called a **dyne**.

NOTE. We may remark that in Statics the unit force generally chosen is *the weight* of 1 lb. In theoretical Dynamics, since we take as the unit mass the mass in which unit force produces unit acceleration, our unit mass (supposing the unit force to be the weight of 1 lb.) would in this case be g lbs.; it will be seen that in Statics we are not concerned with the idea of Mass as that which has *inertia*, but only as that which has weight. Therefore the dynamical unit of mass does not concern us in Statics. It would be very inconvenient to have g lbs. as our theoretical unit of mass, because g is a number whose value is different for different latitudes.

SECTION II.

MOTION IN ONE PLANE.

CHAPTER VI.

DIRECTION.

76. Distance has **direction**.

We have hitherto supposed all our distances to be measured in the same direction, (or in exactly opposite directions), so that it was only necessary to consider the *magnitude* and *sign* of our distances.

In what follows we shall always consider the *direction* of a distance to be an essential part of it. As a necessary consequence we must consider *direction* to be an essential part also of all quantities which vary directly as distance, such as *velocity*, *acceleration*, *force*, *impulse*.

77. Quantities of which direction is an essential part, must be carefully distinguished from those quantities in which it is not.

Quantities having direction are called **vector** quantities.

Distance, *velocity*, *acceleration*, *force*, *impulse* have direction.

Quantities without direction are called **scalar** quantities. *Time*, *mass*, *volume*, *speed* have no direction.

78. It is convenient to use the word **speed** to denote the *magnitude* only of a velocity; just as we use the word *length* to denote the *magnitude only* of a distance. The **average speed** of a point moving in a *curved* line is that which varies directly as the *length* of that portion of the curved line passed over by the point, and inversely as the interval occupied.

The average *speed* of a point moving in a *straight* line is the same as its average *velocity*.

When we speak of a train moving at the rate of 40 miles an hour for a certain interval, we mean that its *average speed* is 40 miles per hour.

A point may have a constant speed while its velocity is changing (in direction).

Similarly we might use the word **quickening** to denote the *magnitude only* of an acceleration.

79. PROP. Any quantity which has direction and magnitude may be represented in a diagram by a finite straight line with an arrow-head.

For the direction of the quantity can be represented by the *direction* of the line with the arrow-head, and the numerical measure of the quantity can be indicated by that of the *length* of the line.

When a distance in a diagram is referred to by the two letters at its extremities, the direction of the arrow-head may be indicated by *the order of the letters*.

80. A distance can always be found which represents the *combined effect*, or the *resultant*, *of two given distances in different directions*.

The combined effect of, (or the result of adding together) two non-directional, (or scalar) quantities is the arithmetical sum of their measures.

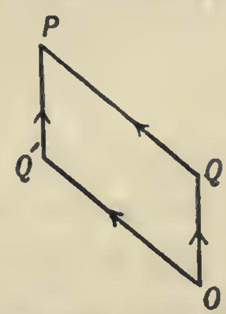
The same is true of two directional, (or vector) quantities which have the same direction.

But the combined effect of two directional, (or vector) quantities having different directions requires definition.

Example. A point P is said to have a distance of 2 inches northwards from a given point O, and *also* a distance of 3 in. towards the north-west from O, when the position of P is found as follows:

Starting from O measure OQ 2 in. towards the north; then *from* Q measure QP 3 in. towards the north-west.

Or thus; starting from O measure OQ' 3 in. towards the north-west and then *from* Q' measure $Q'P$ 2 in. towards the north.



It will be seen that the resulting position of P is the same in each case; for OQ, OQ' are sides of a parallelogram and P is at the corner opposite to O.

81. We may illustrate the meaning of a point having two **simultaneous distances** from a starting point thus:—

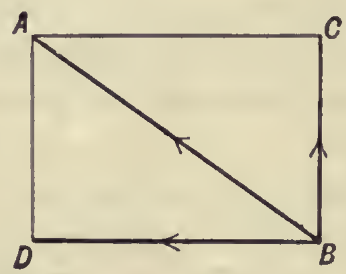
Suppose a sheet of glass to be placed on paper and the point P to be a moveable mark placed on the *glass*.

Let the mark be moved a distance of 3 in. on the glass towards the north-west, and let the glass be moved relatively to the paper, a distance of 2 in. northwards.

It will be seen that the resulting position of the point P with reference to the paper is that found as in the Example in Art. 80.

And it makes no difference to the ultimate position of P whether these distances are passed over *simultaneously*, or whether they are passed over *separately*, one of the movements being finished before the other is made.

Example i. *A point* A *is distant* 3 miles *towards the* North *and* 4 miles *towards the* West *from a point* B; *what is the* total distance *of* A from B?



Draw BC northwards 3 units of length; draw BD westwards 4 units of length; complete the parallelogram $CBDA$;

Then BA represents the required total distance.

By construction, $CBDA$ is a right-angled parallelogram,

therefore $$AB^2 = AD^2 + BD^2. \qquad \text{[Euc. I. 47.]}$$

Let AB contain x units of length; then, by the above statement,

$$x^2 = 3^2 + 4^2 = 9 + 16 = 25;$$

therefore $$x = 5.$$

Thus the distance AB contains 5 units of length.

Each unit in the above diagram represents a mile.

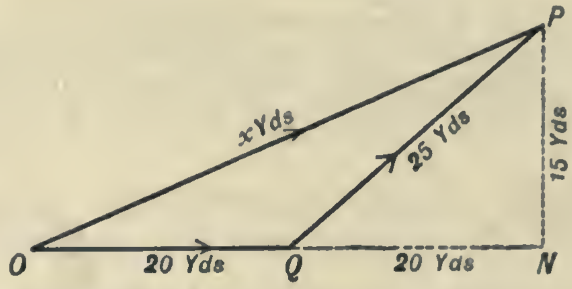
Hence it follows that the total distance of A from B is 5 miles.

And the direction is such that if AB make an angle α with BD,

then, $$\tan \alpha = \frac{DA}{BD} = \frac{3}{4}.$$

Example ii. *A man walks* 20 yards eastwards *and then* 25 yards *in a direction making an* angle whose sine is $\frac{3}{5}$ *with the line pointing eastwards and on the north side of it; find the* direction and magnitude of the resultant distance *of the man from his starting point.*

Let OQ represent a distance of 20 yards eastwards; produce OQ to N; let QP represent the distance of 25 yds., QP being drawn so that $\sin PQN = \frac{3}{5}$.



Draw PN perpendicular to OQ produced; then,

$\frac{NP}{QP}$ is the sine of the angle which QP makes with ON; therefore $\frac{NP}{QP} = \frac{3}{5}$.

Let PN contain y yds. Then $y = \frac{3}{5}$ of $25 = 15$.

Similarly, QN may be shewn to contain 20 yds.

Let OP contain x yds. Then since $OP^2 = ON^2 + NP^2$, [Euc. I. 47]

therefore, $x^2 = (20+20)^2 + (15)^2 = 1600 + 225$

$= 1825$;

therefore, $x = \sqrt{(1825)} = 42{\cdot}7\ldots$

Hence, the required distance is 42·7 yds. Also

$$\tan PON = \frac{NP}{ON} = \frac{15}{40};$$

therefore its direction makes an angle whose tangent is $\frac{15}{40}$, or $\frac{3}{8}$, with the line towards the East.

EXAMPLES. XXIII.

1. A man walks 300 yards northwards and then 400 yards westwards; another man starting from the same place walks 400 yards westwards and then 300 yards northwards; shew that the two men are then at the same total distance from the starting point; and find that distance.

2. A dog on an ice-floe goes 3 miles northwards while the floe itself travels 4 miles eastwards; how far is the dog then from the place on the earth's surface from which he started?

3. A fly crawls along the floor of a railway carriage a distance of 2 ft., at right angles to the line of rails, and at the same time the carriage moves a distance of 10 ft.; find approximately the distance moved over by the fly.

4. A man swims across a river, and in 40 seconds he swims 20 yards; in the same time the current of the river has carried him 15 yards down stream; how far is he from his starting point after 40 secs.?

5. A fly, in a railway carriage which is moving at the rate of 10 ft. per sec. northwards, crawls at the rate of 5 ft. in 10 seconds across the floor of the carriage towards the north-east; what distance will the fly have gone in 10 seconds?

6. On the floor of a railway carriage which is moving with 4 velos northwards, four flies A, B, C, D start from the same place and crawl each with 3 velos, A Northwards, B Southwards, C Westwards, and D towards the North-West. Find their distances from the starting point after 2 seconds.

* *The following examples require the use of the Trigonometric Ratios.*

7. A man walks 350 yards in a certain direction and then turning his direction through an angle of 120 degrees walks 350 yards more; find the direction and magnitude of his resulting distance from his starting point.

8. A man on the deck of a ship moving due north, walks 20 yards towards the north-east while the ship goes 100 yds.; find the direction and magnitude of the whole distance passed over by the man.

9. The distance of a point B from another A is the sum (i.e. the resultant) of two equal distances of 20 yds. inclined to each other at an angle of 60 degrees; find the distance of B from A.

10. When the wind is due North a certain boat can sail either towards the north-east or north-west; it sails first 2 miles towards the N.E., then 2 miles towards the N.W., then 1 mile more towards the N.E.; find the magnitude and direction of its resulting distance from its starting point.

11. I walk 3 miles northwards, then 2 miles in a direction pointing 30^0 to the east of north; what will then be my distance from my starting point?

12. I walk 3 miles in a straight line, I then change my direction through an angle whose sine is $\frac{3}{5}$ and walk 5 miles straight on, I then *turn* (in the same direction as before) through a right angle and walk 1 mile; how far shall I then be from the place from which I started?

82. DEF. The **resultant** of two *distances* from a given point is the one *distance* which is the combined effect or sum of the two distances, taking into account their direction as well as their magnitude.

83. PROP. **The parallelogram of Distances.** *The resultant of two given distances* OA, OB *is* OR, *the diagonal of the parallelogram* AOB.

This follows because the opposite sides of a parallelogram are equal.

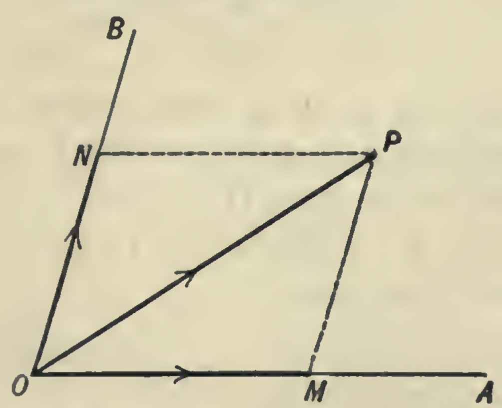
In the figure below, the distance OP is the resultant of the distances OM and MP; that is, of OM and ON, for MP is equal and parallel to ON.

84. When the two distances are in the same direction their resultant, found by the *parallelogram of Distances*, coincides with the two distances placed end and end; so that its magnitude is their *sum*.

When the two distances are in opposite directions the magnitude of their resultant is their *difference*.

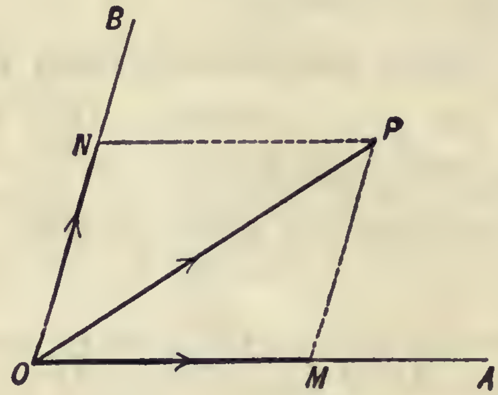
85. It is often convenient to express the distance of one point from another as the resultant of two distances in given directions.

Example. Express the distance OP *as the resultant of two distances in directions parallel to* OA *and* OB *respectively.*



Through P draw PM and PN parallel to OB and OA respectively. Then the distance OP is the resultant of the distances OM and ON.

86. DEF. When the one distance OP is expressed as the sum of two others OM and ON, the two distances OM and ON are called the **components** of the distance OP;



and the distance OP is said to be **resolved** into its two *components* OM and ON in those directions.

The distance OP is clearly the *resultant* of its two components OM and ON.

It is usually convenient to take the two component directions **at right angles** to each other.

DEF. When the two components are mutually at right angles, each component is called **the** *resolved part* of the resultant in its own direction.

The **most convenient method of finding the resultant of a number of distances** (or other directional quantities) is by taking their resolved parts in two directions mutually at right angles. Thus,

Example i. *The road from* A to B *runs in a straight line for* 2 miles *from* A *to* H; *then after turning through an angle of* 30°, *it runs in a straight line for* 3 miles *from* H *to* K; *then after turning through an angle of* 15°, *runs in a straight line for* 1 mile *from* K to B; *how far is it from* A *to* B *as the crow flies?*

Resolve the distance HK into its two resolved parts HN and $HL=NK$ along and perpendicular to AH.

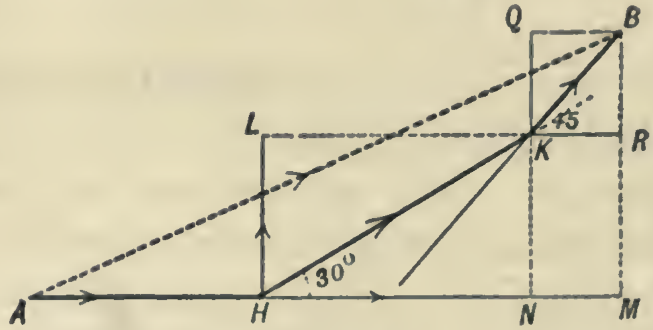
Resolve the distance KB into its two resolved parts $KR=NM$ and $KQ=RB$ along and perpendicular to AH.

Then, $HN = HK \cos 30 = \frac{1}{2}\sqrt{3} \times 3$ miles,

$NK = HK \sin 30 = \frac{1}{2} \times 3$ miles,

$NM = KR = KB \cos 45^0 = \frac{1}{\sqrt{2}} \times 1$ mile,

$RB = \quad KB \sin 45^0 = \frac{1}{\sqrt{2}} \times 1$ mile.



Hence, the number of miles in AM is $2 + \frac{1}{2} \times 3\sqrt{3} + \frac{1}{2}\sqrt{2}$.
The number of miles in MB is $\frac{1}{2} \times 3 + \frac{1}{2}\sqrt{2}$.
Therefore, the number of miles in AB is

$$\sqrt{\{(2 + \tfrac{1}{2} \times 3\sqrt{3} + \tfrac{1}{2}\sqrt{2})^2 + (\tfrac{3}{2} + \tfrac{1}{2}\sqrt{2})^2\}}$$
$$= \sqrt{\{(2 + \tfrac{1}{2} \times 3 \times 1{\cdot}7320\ldots + \tfrac{1}{2} \times 1{\cdot}4142\ldots)^2 + (1{\cdot}5 + \tfrac{1}{2} \times 1{\cdot}4142\ldots)^2\}}$$
$$= \sqrt{\{(5{\cdot}305\ldots)^2 + (2{\cdot}207\ldots)^2\}} = \underline{5{\cdot}746\ldots \text{ nearly.}}$$

Example ii. *The distance of* B *from* A *is the* resultant *of the three distances* AP, AP′, AP″ *of* 2 miles, 3 miles and 1 mile *respectively where the* angle PAP′ = 30°, *and the* angle P′AP″ = 15°; *what is the* distance of B from A?

It will be found that this is only another way of stating *Example* i.

EXAMPLES. XXIV.

1. Resolve a distance of 100 yards into two components mutually at right angles one of which makes an angle of 30° with it.

2. Resolve a distance of 25 yards into two components mutually at right angles one of which makes an angle whose sine is $\frac{3}{5}$ with it.

3. A man walks 200 yards in a straight line, then turns through an angle of 45° and walks 150 yards further in a straight line; find his resultant distance from the starting point.

4. Find the resultant of two distances of 300 yards and 450 yards making an angle of 45° with each other.

5. I walk 1 mile towards the north-east; what is the *resolved part* of this distance in the northerly direction? what is the *resolved part* of this distance towards the north-west?

6. Find the resolved part of a distance of 360 yards in the direction which makes an angle of 60° with it.

87. When a point has a given uniform velocity u velos in a given direction, then after an interval t seconds the point is at a certain *distance* from its initial position; this distance is ut feet, and is in the direction of the velocity. This distance is called **the distance due to the velocity**.

88. A point is said to have **simultaneous** velocities and accelerations when its distance after an interval from its initial position is the resultant of the distances due to those velocities and accelerations separately.

Example. A point has -32 celos *vertically upwards, and at a certain instant it has* 200 velos *vertically upwards, and* 40 velos *in a horizontal direction; find its position after* 3 seconds.

Let O be its initial position.

Draw ON horizontally to represent 3×40 ft.

From N draw NQ vertically upwards to represent 3×200 ft. and from Q draw QP vertically downwards to represent $\frac{1}{2} \times 32 \times 9$ ft.

Then P represents the position of the point after 3 seconds.

EXAMPLES. XXV.

Find the position of a point having the following;

1. 3 velos northward and 4 velos eastwards, after 30 seconds.

2. 2 velos southwards and 3 velos eastwards, after 15 secs.

3. 4 velos northwards and 7 velos southwards, after 8 secs.

4. 20 velos northwards, 15 velos eastwards and 15 velos northwards, after 8 secs.

5. 20 velos northwards and -3 celos westwards, after 10 seconds.

6. Three velocities of 20 velos each making an angle of 120^0 with each other, after 10 seconds.

7. 32 celos westwards, 20 velos northwards, initially after 2 secs.

8. Three accelerations, 15 celos westwards, 20 celos northwards and 15 celos eastwards after 1 second, initially at rest.

89. A particle which having been projected from any point in any direction then moves under the action only of its own weight is called a **projectile.**

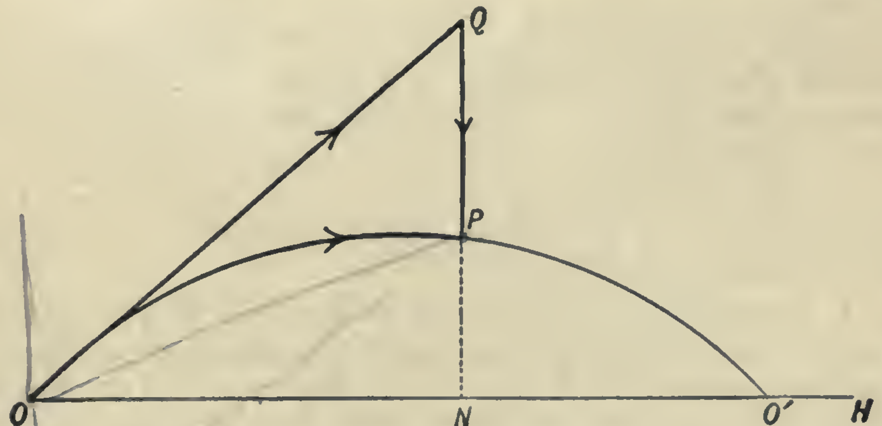
The motion of a **projectile** affords an important example of a point which after any interval has two simultaneous distances from its initial position.

Example i. *A particle is projected with* u velos *in a direction making an* angle α *with the horizontal plane; find its position after* t seconds (*neglecting the resistance of the air*).

It is understood that the only force acting on the particle is its own weight, which acts vertically downwards.

By Art. 33 this force produces in the mass the same acceleration whether the particle has other velocity or not.

Hence, the *distance* of the particle after t seconds from its initial position is the resultant of two distances, one due to the u velos, the other due to the acceleration g celos vertically downwards.



Let O be the initial position of the particle P; let OH be a horizontal line; then we may find the position of P after t seconds thus:

Draw OQ, making an angle α with OH, so that OQ represents a distance of ut feet;
from Q draw QP vertically downwards, so that QP represents $\frac{1}{2}gt^2$ feet.

Then P is the position of the particle after t seconds.

For OQ is the distance due to the velocity u velos, and QP is the distance due to the acceleration g celos†.

Example ii. *A stone is thrown from a point* O, *which is distant* 100 yds. *from* Q *the top of a tower, with the velocity* 100 velos *in the direction* OQ; *find where the stone will strike the tower.*

In the above figure let QN be the tower, O the point of projection; then $OQ = 100$ yds. $= 300$ feet.

† The student is advised to find the position of the projectile at short intervals, by drawing the lines representing its distances carefully to scale; he will in that way trace out the *path* of the projectile.

If gravity were not acting the stone would go straight from O to Q with 100 velos in 3 seconds.

Since gravity acts, the stone describes the curved path from O to P; at the end of 3 seconds its position is found by drawing $OQ = 300$ ft. and then QP vertically downwards so that $QP = \frac{1}{2} \times 32 \times 3^2$ feet $= 144$ feet.

Thus the stone will strike the tower 144 feet below the point Q.

EXAMPLES. XXVI. a.

Find the positions of the three following projectiles after intervals of 1, 2 and 3 seconds respectively from the instant of projection;

1. of a stone thrown from the top of a cliff in a horizontal direction with 100 velos.

2. of a stone thrown at an angle of 45 degrees to the horizon with 100 velos.

3. of a bullet projected at an angle of 30^0 to the horizon with 1000 velos.

4. A rifle is pointed at the top of a tower 100 ft. high standing on a horizontal plane, the top being at a distance of 800 yards from the rifle. What must be the velocity of projection that the bullet may strike the foot of the tower?

5. A rifle is pointed at the top of a tower which is distant 1000 yards from the rifle; the bullet is projected with 800 velos; where will it strike the tower?

6. A vessel sailing southwards at the rate of $7\frac{1}{2}$ miles an hour points a gun horizontally at a vertical rock whose face, distant $\frac{1}{4}$ mile from the ship, is parallel to the path of the vessel; if the muzzle velocity of the shot is 1320 velos, how many feet from the spot at which the gun was pointed is the spot at which the shot will strike the rock?

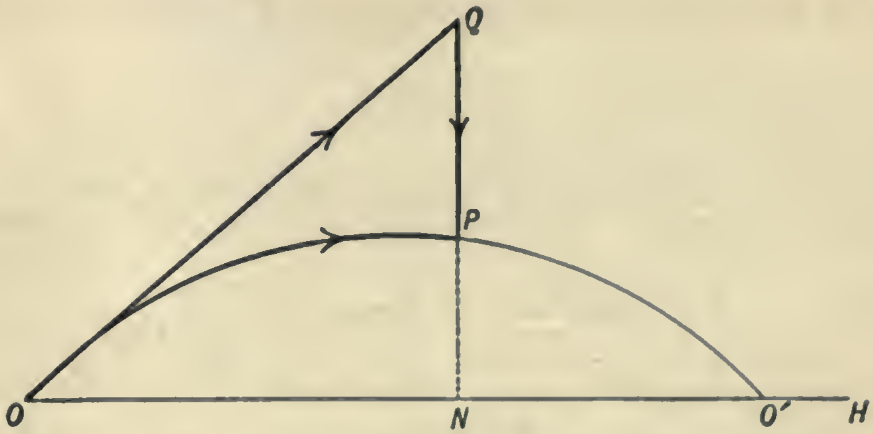
THE RANGE AND TIME OF FLIGHT.

90. DEF. When a particle is projected with a given velocity under the action of its own weight, its **range** is the horizontal distance from its initial position of the point at which the particle strikes the ground;

its **time of flight** is the interval occupied between its start and the instant at which it strikes the ground.

91. PROP. *To find* (i) *the* range, (ii) *the* time of flight, *and* (iii) *the* greatest height *of a particle projected from a point on a horizontal plane.*

Let u velos be the initial velocity; α the angle of projection with the horizontal plane.



Draw $OQ = ut$ feet, making an angle $HOQ = \alpha$, with the horizontal line OH; draw $QP = \frac{1}{2}gt^2$ feet vertically downwards; then P is the position of the particle after t seconds.

Produce QP to cut OH in N.

Then, $\qquad ON = OQ \cos \alpha = ut \cos \alpha$ feet.

Also $\qquad NP = NQ - PQ = OQ \sin \alpha - PQ$

$$= (ut \sin \alpha - \tfrac{1}{2}gt^2) \text{ feet.}$$

When P strikes the ground NP vanishes; hence, if t seconds be the time of flight,

$$ut \sin \alpha - \tfrac{1}{2}gt^2 = 0,$$

that is, either $\qquad t = 0, \quad \text{or,} \quad t = \dfrac{2u \sin \alpha}{g}.$

Hence, when P reaches the ground *again* at O'

$$t = \frac{2u \sin \alpha}{g}.$$

Therefore the range, OO', is equal to ON [which $= (ut \cos \alpha)$ ft.] when $t = \dfrac{2u \sin \alpha}{g}$. Hence the range

$$= \frac{2u \sin \alpha}{g} \times u \cos \alpha \text{ ft.} = \frac{2u^2 \sin \alpha \cos \alpha}{g} \text{ feet,} \qquad \text{(i)}$$

and the time of flight is $\dfrac{2u \sin \alpha}{g}$ seconds. $\qquad$ (ii)

Also, after t seconds the height $NP = (ut \sin\alpha - \frac{1}{2}gt^2)$ feet

$$= \frac{2ugt \sin\alpha - g^2t^2}{2g} \text{ feet}$$

$$= \frac{u^2 \sin^2\alpha - (u^2 \sin^2\alpha - 2gut \sin\alpha + g^2t^2)}{2g} \text{ feet}$$

$$= \left[\frac{1}{2}\frac{u^2 \sin^2\alpha}{g} - \left(\frac{u \sin\alpha}{g} - t\right)^2 \frac{g}{2}\right] \text{ feet};$$

this is *greatest* when $t = \dfrac{u \sin\alpha}{g}$; that is, at the *middle* of the time of flight; and then $NP = \frac{1}{2}\dfrac{u^2 \sin^2\alpha}{g}$ feet, which is the greatest height. [See also Art. 125.] (iii)

Example. A projectile has initially 1600 velos *in a direction making an angle* whose sine is $\frac{1}{10}$ *with a horizontal plane; find the* time of flight, *the* range, *and the* greatest height.

Here, $\sin\alpha = \frac{1}{10}$

and $\cos\alpha = \sqrt{(1 - \sin^2\alpha)} = \sqrt{\frac{99}{100}} = \frac{1}{10}\sqrt{99}$

$= \cdot 9949\ldots$

The vertical distance after t seconds is

$$(ut \sin\alpha - \tfrac{1}{2}gt^2) \text{ feet};$$

that is, $(1600 \times \frac{1}{10} \times t - 16t^2)$ feet.

This is zero when $t = 10$.

Hence, the time of flight is 10 seconds (i).

The horizontal distance after t seconds is $ut \cos\alpha$ feet.

Hence, the range is $1600 \times 10 \times \cdot 9949\ldots$ that is, about 15918 feet (ii).

The greatest height is the vertical distance at the middle of the time of flight;

that is, $(1600 \times 5 \times \frac{1}{10} - 16 \times 25)$ feet.

Hence, the greatest height is 400 feet (iii).

EXAMPLES. XXVI. b.

Find the range, the time of flight and the greatest height when a particle is projected from a point on a horizontal plane under the action only of its weight; the direction of projection making the following angles with the horizon and the initial velocity being as follows.

1. Angle, 45^0; velocity, 100 velos.

2. Angle, 45^0; velocity, 50 velos.

3. Angle, 45^0; velocity, 64 velos.

4. Angle, 30^0; velocity, 250 velos.

5. Angle, 60^0; velocity, 100 velos.

6. Angle, 60^0; velocity, 50 velos.

7. The angle whose sine is $\frac{3}{5}$; velocity 75 velos.

8. The angle whose sine is $\frac{4}{5}$; velocity 125 velos.

9. The angle whose sine is $\frac{1}{100}$; velocity 1000 velos.

10. The Martini-Henry rifle gives its bullet a muzzle velocity of 1,315 velos; find its range when it is elevated to an angle whose sine is $\frac{1}{120}$, neglecting the resistance of the air.

11. The new Enfield-Martini rifle gives its bullet a muzzle-velocity of 1570 velos; find its range when it is elevated to an angle whose sine is $\frac{1}{130}$, neglecting the resistance of the air.

12. Find the height of the bullet in Questions **10** and **11** at the middle of the flight.

13. The 110 ton gun gives its shot of 1800 lbs. (when using 950 lbs. of Waltham Powder) a muzzle velocity of 2100 velos; find its range on a horizontal plane when elevated to an angle whose sine is $\frac{1}{10}$.

14. The 110 ton gun gives a muzzle velocity of 2128 velos when using 1000 lbs. of powder; find the range on a horizontal plane when the elevation is an angle whose sine is $\frac{1}{5}$.

CHAPTER VII.

THE PARALLELOGRAM OF VELOCITIES.

92. DEF. The **resultant** of two simultaneous *velocities* is a *velocity* such that the distance due to it after any interval is equal to the resultant of the two distances due to the two simultaneous velocities.

93. The two simultaneous velocities are called the *components* of the resultant; and when they are mutually *at right angles* they are each called the *resolved part* of the resultant in their own direction.

94. In fact it will be found that all definitions and propositions applicable to distances may be applied to velocities (and also to any *vector* quantity). Thus we have

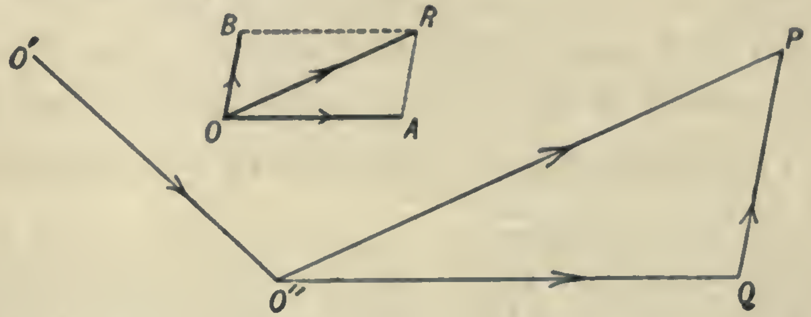
95. PROP. **The parallelogram of Velocities.** *When two velocities are represented (in direction and magnitude) by two straight lines* OA *and* OB, *their resultant is represented (in direction and magnitude) by* OR *the diagonal of the parallelogram* AOB.

Let a point P have (in addition to any other velocity) the two simultaneous velocities represented by OA and OB.

We have to prove that OR represents the resultant of the two simultaneous velocities represented by OA, OB.

That is, we have to shew that the position of a point Π which has the same motion as P, except that it has the velocity OR instead of the two simultaneous velocities OA, OB is, after any interval, the same as that of P.

Let OA represent u' velos; let OB represent u'' velos; and let OR represent u velos.



Let O' represent the position of the moving point at the beginning of an interval t seconds.

Let $O'O''$ represent the distance due to all the other velocities of the moving point for the interval t seconds

Draw $O''Q$ parallel to OA to represent $u't$ ft.; draw QP parallel to OB to represent $u''t$ ft. Then P is the position of the moving point at the end of t seconds.

Again, consider the motion of the point Π. It has the same motion as P, except that the velocities OA and OB are replaced by the velocity OR. Hence we find the position of Π after t seconds by drawing $O'O''$ as before and then from O'' drawing a line $O''\Pi$ parallel to OR to represent ut ft.

We have to shew that Π will then coincide with P.

That is, we have to shew that $O''P$ is parallel and proportional to OR.

Now
$$\frac{O''Q}{QP} = \frac{u'}{u''} = \frac{OA}{OB} = \frac{OA}{AR};$$
and $O''Q$, QP are parallel to OA and AR respectively.

Therefore [Euclid VI. 6] $O''P$ is parallel and proportional to OR.

Thus, it makes no difference to the position of P after any interval when we replace the two simultaneous velocities OA and OB by the velocity OR. Q. E. D.

96. When the moving point has only the two velocities OA and OB, then in Art. 95, $O'O''$ disappears, and we see that since $O''P$ is always parallel to OR (which is fixed in direction and magnitude) and always represents ut ft., the actual motion of the moving point is the velocity OR.

97. The resultant of a number of simultaneous velocities is a uniform velocity.

For any two of them may be replaced by their resultant; this resultant and a third velocity may be replaced by their resultant; and so on.

PROP. *The actual motion of a point which is moving with a number of simultaneous uniform velocities only is uniform velocity.*

For its resultant velocity, given by the parallelogram of velocities, is always the same; and its resultant distance from its initial position is always parallel and proportional to this resultant velocity.

Its actual path is therefore a straight line along which it moves with uniform velocity.

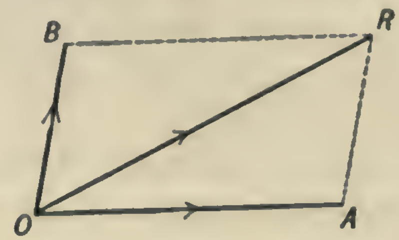
98. PROP. *When a point is moving with given variable velocities* **its actual velocity at any instant** *is the resultant of the velocities which it has at that instant.*

For, consider another point Q moving with uniform velocities such that at that instant it has the *same* simultaneous velocities as the point P, then Q will, at that instant, be moving in the same direction with the same speed as the first point P.

99. The Parallelogram of Velocities is sometimes stated in the following form, which is called

The Triangle of Velocities. When a point has two simultaneous velocities, the resultant velocity may be found as follows: Draw a line OA to represent one of the velocities; from A draw a line AR to represent the other velocity; the third side OR of the **triangle** OAR represents the resultant velocity of the two simultaneous velocities OA, AR.

This proposition is only another way of stating the parallelogram of velocities.



For in the parallelogram $OARB$ it is clear that if OA, OB represent two simultaneous velocities, then AR represents a velocity equal and parallel to that represented by OB.

100. *To find the resultant of any number of velocities represented by straight lines* OA, OB, OC,...

I. First find OR' the resultant of OA and OB; then we find OR'' the resultant of OR' and OC; and so on.

Or II. Draw from A a line AR' parallel and equal to OB; from R' a line $R'R''$ parallel and equal to OC; and so on: the line joining O to the last point R represents the resultant of all the velocities.

III. **Practically** as follows; See Art. 86.

Resolve each velocity into its resolved parts in two convenient directions, mutually at right angles; add the velocities in each of those directions and find the resultant of the two sums.

Example i. The student is advised to use the figure and work of Example i. p. 80 in the following.

Find the resultant of the velocities 20 velos, 30 velos *and* 10 velos, *the angles between their direction being* 30^0 and $15°$ *respectively.*

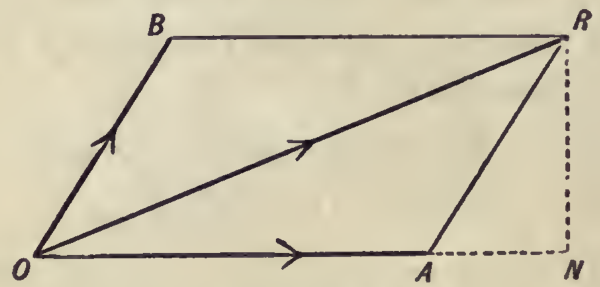
Example ii. *A stone is thrown with* 33 velos *from a train, at right angles to its direction of motion, while the train is going* 30 miles an hour; *find the* initial velocity *of the stone relatively to the ground.*

Relatively to the ground the stone has two simultaneous velocities of 33 velos and 44 velos respectively, at right angles to each other.

Therefore, by the parallelogram of velocities, their resultant is a velocity of 55 velos making with the direction of the train an angle whose tangent is $\frac{3}{4}$.

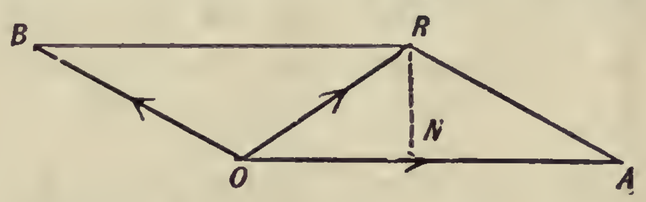
Example iii. *A carriage is going at the rate of* 15 miles an hour; *a man jumps off, giving himself an additional horizontal velocity of* 10 miles an hour; *find his* velocity relative to the ground *when he jumps in a direction making* (i) an angle of 60^0, (ii) an angle of 150^0, *with the direction in which the carriage is going.*

Draw the parallelogram of velocities $AOBR$; draw RN perpendicular to OA; then OR is the required velocity relative to the ground. Then $$OA=15,\ OB=AR=10.$$



In case (i) $$NAR=AOB=60^0;$$
$$AN=AR\cos 60^0=\tfrac{1}{2}\times 10;$$
$$NR=AR\sin 60^0=\tfrac{1}{2}\sqrt{3}\times 10;$$
therefore, $$OR^2=(OA+AN)^2+NR^2$$
$$=(15+5)^2+(\sqrt{3}\times 5)^2$$
$$=475=(21\cdot 8\ldots)^2.$$

Therefore the velocity required is 21·8... miles per hour.



In case (ii) $$NAR=30^0;\ NA=AR\cos 30^0=\tfrac{1}{2}\sqrt{3}\times 10;$$
$$NR=AR\sin 30^0=\tfrac{1}{2}\times 10;$$
therefore, $$OR^2=(OA-NA)^2+NR^2$$
$$=(15-5\sqrt{3})^2+5^2$$
$$=(15-5\times 1\cdot 7320\ldots)^2+25$$
$$=(8\cdot 18\ldots)^2.$$

Therefore the velocity required is 8·18...miles per hour.

NOTE. When the resultant velocity of a point is zero, the point is at rest.

For the *distance* due to all its velocities after any interval is also zero.

101. When a point at one instant has a certain velocity and at another instant has a different velocity, then some velocity must have been added to its original velocity in the interval.

The velocity which has been added may be found by the parallelogram of velocities; for the new velocity is the resultant of the original velocity and of the velocity added.

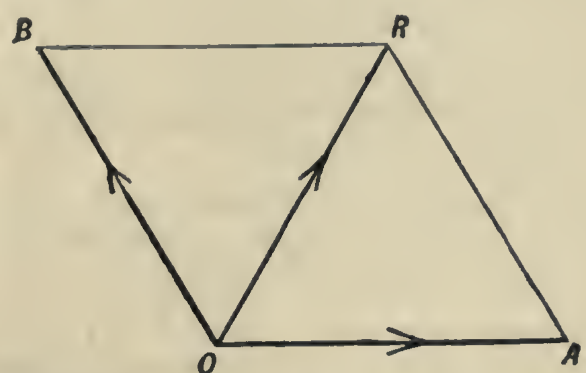
Example. A point in a certain interval has its velocity changed from 20 velos *in a given direction to* 20 velos *in a direction making an* angle 60^0 *with the first direction; find the* additional velocity.

Let OA represent the original velocity; let OR represent the new velocity; join RA; draw OB parallel and equal to AR.

Then OB represents the additional velocity required.

The angle $AOR = 60^0$ and $OA = OR$, therefore AOR is an equilateral triangle; hence the angle $ROB = ORA = 60^0$.

Therefore $OB = OA$.



Thus, the additional velocity is a velocity of 20 velos in a direction making an angle 120^0 with the original velocity.

EXAMPLES. XXVII.

Find the resultants of the following 12 pairs of velocities:

1. 12 velos and 16 velos mutually at right angles.

2. 20 velos and 15 velos mutually at right angles.

3. 30 miles an hour and 40 miles an hour at right angles to each other.

4. 2 velos and 8 velos mutually at right angles.

5. a velos and b velos at right angles to each other.

6. a ft. per second and m miles per hour mutually at right angles.

7. 3 velos and 3 velos inclined at an angle of 120^0.

8. a velos and a velos inclined at an angle of 60^0.

9. 3 velos and $3\sqrt{2}$ velos inclined at an angle of 45^0.

10. 10 velos and 20 velos inclined at an angle of 60^0.

11. 40 velos and $20\sqrt{3}$ velos inclined at an angle of 150^0.

12. 20 velos and 25 velos inclined at an angle whose sine is $\frac{3}{5}$.

13. Find the distance in 10 seconds due to 3 simultaneous velocities of 2 velos, 3 velos and 1 velo making the two angles of 30^0 and 15^0 with each other.

14. A particle is given a velocity of 50 velos in a direction making an angle 45^0 with the horizon; resolve this velocity into two; one horizontal and the other vertical.

15. A particle has velocity 100 velos making an angle whose sine is $\frac{1}{13}$ with the horizon; what is the resolved part of the velocity in a horizontal direction?

16. From a ship sailing at the rate of 25 miles an hour a bullet is fired with a muzzle velocity of 1100 feet per second in a direction at an angle whose sine is $\frac{1}{13}$ with the direction of the ship's motion; find the initial velocity of the shot relative to the water.

17. A football moving with 20 velos receives a kick, after which its direction is changed through a right angle but its speed is unaltered; what velocity was given it by the kick?

18. A particle travelling northwards with 20 velos, receives an additional velocity of 20 velos so that the resultant velocity is 20 velos; in what direction was the additional velocity?

19. The direction of a point's motion is altered by 45^0 while its speed is unchanged; what is the alteration in its velocity?

20. A cricket ball having 30 velos is struck by a bat so that its velocity is changed to 40 velos in a direction making an angle 150^0 with its former direction; what velocity was given to the ball by the bat?

21. A stone thrown from a train has a certain velocity given it at right angles to the path of the train, so that relatively to the ground it has 60 velos in a direction making 60^0 with the path of the train; find the velocity of the train.

22. A particle is projected with a velocity whose vertical and horizontal components are 50 velos and 10 velos respectively; the vertical velocity is diminished at the rate of 32 velos per second, the horizontal velocity is unaltered; find (the magnitude and direction of) the velocity of the particle after 1 second. What was the direction in which it was projected?

23. A spherical shot is rolling directly across the smooth horizontal deck of a ship with a velocity 10 velos; find where it would strike the side of the ship supposing the ship, which is going 10 miles an hour, to be suddenly stopped when the shot is 20 ft. from the side.

24. A balloon rising vertically with a velocity of 10 miles an hour is carried by the wind over a horizontal distance of 100 yds. in 20 seconds; find the velocity of the balloon.

25. A man tricycling westwards at the rate of 8 miles an hour, feels the wind to be blowing from the South; on increasing his speed to 16 miles an hour the wind appears to be blowing from the South-West; find the velocity of the wind.

26. Light coming from a star vertically over-head is travelling at the rate of $186,000 \times 1760 \times 3$ velos; an observer at the equator has a horizontal velocity due to the rotation of the earth of say 25,000 miles per 24 hours; in what direction will the light appear to be coming?

27. Rain drops are falling vertically with 40 velos, but they are also carried horizontally with the wind so that they appear to fall at an angle 30^0 with the vertical; what is the velocity of the wind?

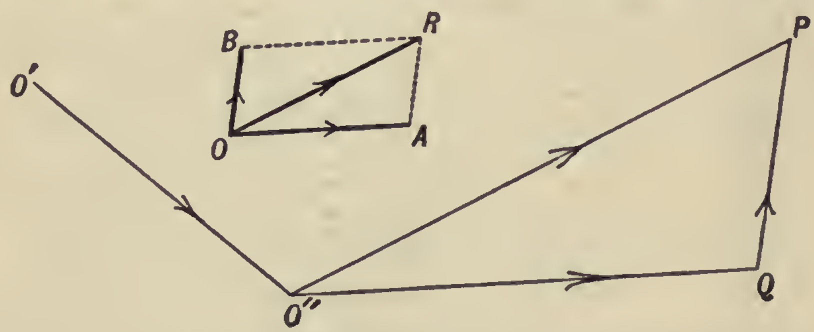
28. Walking against the wind and rain of Question **27** I find that when holding a straight tube at an angle of 45^0 to the vertical, the rain passes through the tube without touching it; at what rate am I walking?

THE PARALLELOGRAM OF ACCELERATIONS.

102. DEF. The **resultant** of two simultaneous *accelerations* is an acceleration, such that the distance due to it after any interval is the resultant of the distances due to the two accelerations.

103. **The parallelogram of accelerations.** *When two simultaneous accelerations are represented (in direction and magnitude) by two straight lines* OA, OB, *their resultant is represented (in direction and magnitude) by* OR, *the diagonal of the parallelogram* AOB.

Let a point have (besides any other velocities) the two simultaneous accelerations represented by OA and OB.



We have to shew that the position of a point Π which has the same motion as P except only that it has the acceleration OR instead of the two simultaneous accelerations OA, OB, is after any interval the same as that of P.

Let OA represent α' celos; let OB represent α'' celos; let OR represent α celos.

Let O' represent the initial position of the moving point.

Let $O'O''$ represent the distance due to all the other velocities of the moving point for the interval t seconds.

Draw $O''Q$ parallel to OA to represent $\frac{1}{2}\alpha' t^2$ ft., draw QP

parallel to OB to represent $\frac{1}{2}\alpha'' t^2$ ft. Then P is the position of the point after t seconds.

Again consider the motion of the point Π. It has the same motion as P, except that the accelerations OA, OB are replaced by the acceleration OR. Hence we find the position of Π after t seconds by drawing $O'O''$ as before and then from O'' drawing a line $O''\Pi$ parallel to OR to represent $\frac{1}{2}\alpha t^2$ ft.

We have to shew that Π coincides with P.

That is we have to shew that $O''P$ is parallel and proportional to OR.

Now $$\frac{O''Q}{QP} = \frac{\alpha'}{\alpha''} = \frac{OA}{OB} = \frac{OA}{AR},$$

also $O''Q$, QP are parallel to OA, AR respectively.

Therefore $O''P$ is parallel and proportional to OR.

Thus it makes no difference to the position of P after any interval when we replace the two simultaneous accelerations OA, OB by the acceleration OR. Q. E. D.

104. When a moving point is moving with given **variable** simultaneous accelerations, *its actual acceleration at any instant* is found by the parallelogram of acceleration.

For, consider another point Q moving with uniform accelerations, such that at that instant it has the *same* velocities and accelerations as the first point P; then Q will, at that instant, be moving with the same actual velocity and with the same actual acceleration as the first point P.

105. The resultant of a number of simultaneous accelerations may be found in exactly the same way as the resultant of a number of simultaneous distances or velocities. [See Art. 97.]

106. When the resultant of all the accelerations which a point has is zero, the point is moving with uniform velocity.

For the motion is that of a point having no acceleration.

107. When a point has simultaneous acceleration, each acceleration adds its own velocity to the initial velocity of the point independently of any other velocities which the point may have. Hence

108. The parallelogram of accelerations may be deduced from the parallelogram of velocities in precisely the same words as the parallelogram of velocities was deduced from that of distances by reading velocity for distance and acceleration for velocity in Art. 95.

109. The parallelogram of acceleration may also be stated as the **Triangle of Accelerations** in the words of Art. 99.

EXAMPLES. XXVIII.

1. A man is starting to leave a railway carriage walking directly towards the door with an acceleration 3 celos, and the train itself is stopping with retardation 4 celos; what is the direction of the resultant acceleration of the man?

2. A man in a train which is getting up its speed with an acceleration 5 celos, gives a ball an acceleration 12 celos, in the direction at right angles to the motion of the train; find the resultant acceleration of the ball.

3. Supposing in Question **2** the man commences to throw the ball just as the train is starting and he keeps up the pressure on the ball with his hand (thus producing 12 celos) for 1 second, find the initial velocity of the ball on leaving his hand.

4. Supposing in Question **2** the train is moving at the rate of 30 miles an hour, (and with 5 celos) when the man commences his throw, and that he keeps up the pressure for 1 sec., find the initial velocity of the ball.

5. A man runs across a railway carriage which is 7 ft. across with an acceleration of 5 celos; the carriage is moving with a uniform velocity of 30 miles an hour; find the velocity with which he leaves the carriage.

6. A train is pulling up with a retardation of 10 celos; I wish to walk straight across the carriage with an acceleration of 10 celos; in what direction shall I have to apply a horizontal force to my body in order to do so?

CHAPTER VIII.

THE PARALLELOGRAM OF FORCES.

110. By Arts. 31, 33 every *force* applied to a mass produces in that mass its proper *acceleration* in its own direction independently of any other motion the mass may have.

It follows that every force acting upon a mass for a given interval produces in that mass its own proper velocity in its own direction; in other words that every *impulse* applied to a mass produces in that mass its proper *velocity* in its own direction, independently of any other motion the mass may have.

Illustration. Suppose a man, in a railway train moving with 44 velos westwards, to apply to a mass (a stone) a force northwards for a certain interval. He applies an impulse northwards and therefore gives to the stone a velocity northwards; say 33 velos. The stone has also the velocity of the train. The stone has therefore on leaving the man's hand two simultaneous velocities, one of 44 velos westwards and another of 33 velos northwards. Its motion relative to the earth will be a velocity of 55 velos in a direction between North and West. On leaving the man's hand the stone has at once an acceleration due to gravity of 32 celos downwards as explained in Art. 89.

111. When two forces act simultaneously on a mass they produce two simultaneous accelerations; each force producing in the mass its own proper acceleration in its own direction.

In Art. 103 it is proved that two simultaneous accelerations may be replaced by a single acceleration, hence

DEF. The **resultant** of two *forces* is that *force* which acting on any mass will produce in that mass the resultant of the accelerations which the two forces produce when acting simultaneously on the same mass.

112. PROP. **The parallelogram of Forces.** *When two forces acting on a particle are represented (in direction and magnitude) by the straight lines* OA, OB, *their resultant is represented (in direction and magnitude) by* OR *the diagonal of the parallelogram* AOB.

By Art. 33, when forces act simultaneously on a particle each force produces in the particle an acceleration in its own direction proportional to the force producing it.

Hence forces which act upon the *same* particle may each be represented by that straight line which represents the acceleration which it produces in the particle.

But, by Definition, the *resultant* of two forces is that force which produces in a particle the resultant OR of the two accelerations OA, OB produced in the particle by the forces.

Therefore the resultant of two forces represented by two straight lines OA, OB, is *represented* by OR the diagonal of the parallelogram $AORB$. Q. E. D.

113. PROP. **The Parallelogram of Impulses** follows immediately upon the parallelogram of velocities.

For, since the velocities which any given impulses produce in the same mass are proportional to and in the same direction as the impulses which produce them, it follows that *impulses*, whether simultaneous or not, which act upon the same particle may be represented by the lines which represent the *velocities* they produce.

Therefore the resultant of two impulses OA, OB is represented by OR the diagonal of the parallelogram $OARB$.

114. In considering the motion of a mass acted on by several forces either

I. We may consider that each force produces its own acceleration in the mass, and then find the acceleration of the mass by finding the resultant of all these accelerations, or

II. We may first find the resultant of all the forces, and then consider that the forces are equivalent to this resultant only, which resultant produces the actual acceleration of the mass.

These two ways of considering the motion of a mass are identical in method.

It should be noticed that the three propositions called the Parallelogram of Distances, the Parallelogram of Velocities, and the Parallelogram of Acceleration are *geometrical facts*; while the proposition known as the Parallelogram of Forces is deduced from the Parallelogram of Accelerations, by means of the definition of Force as that which produces acceleration in mass, each force producing its own acceleration exactly as if the others did not exist.

When the resultant of any number of forces is zero, they produce no resultant acceleration; in other words, they have no effect on the motion of any mass on which they simultaneously act.

Forces whose resultant is zero are said to be in **equilibrium**.

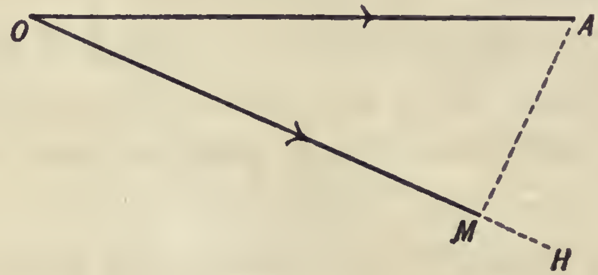
As a particular case we may notice, that when one of three forces is equal and opposite to the resultant of the other two, the three forces are in equilibrium.

115. The following definitions are similar to those of Arts. 86, 93.

The **components** of a given force are two forces of which the given force is the resultant.

The resolved part of a force in a given direction is the component in that direction, the two components being **at right angles** to each other.

Example. Find the resolved part *of a force of* 30 lbs. weight *in the direction making an angle* 30^0 *with it.*



Let OA represent the force of 30 lbs. weight; let OH be the given direction so that $HOA = 30^0$.

From A draw AM perpendicular to OH.

Then OM is the resolved part of OA in the direction OH. [For if the right-angled parallelogram $MOAN$ be completed OM and ON are the rectangular components of OA.]

Now $$\frac{OM}{OA} = \text{cosine } 30^0 = \tfrac{1}{2}\sqrt{3};$$

$$\therefore\ OM = \tfrac{1}{2}\sqrt{3} \times OA.$$

116. The most convenient method of finding the resultant of given forces in one plane, acting simultaneously upon a particle, is by resolving each force into its resolved parts in two convenient rectangular directions. See Arts. 86, 100.

Example i. *Find the* resultant *of forces of* 8 poundals, 16 poundals, 10 poundals *and* 10 poundals *represented by lines* OA, OB, OC, OD *such that the angle* $AOB = 45^0$, $AOC = 60^0$, $AOD = 120^0$.

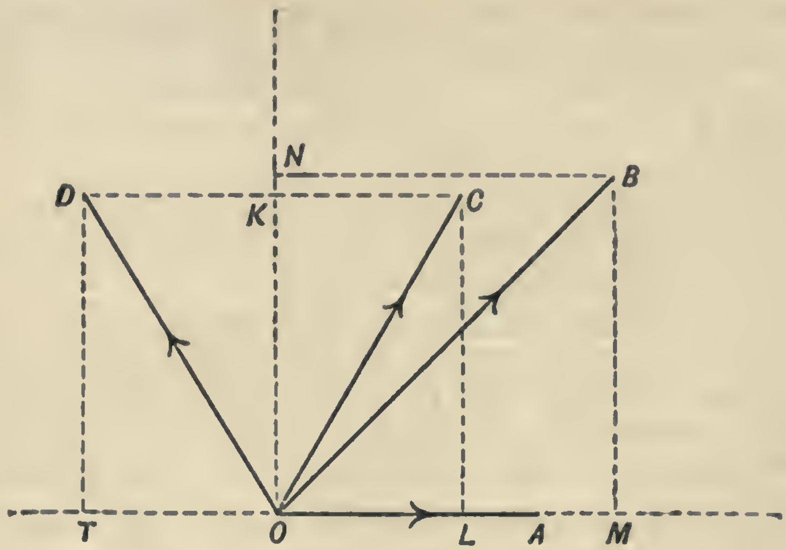
Choose OA and the line ON perpendicular to OA as the two rectangular directions in which to resolve the forces.

Resolve OB into OM and ON. Then

$$OM = OB \cos 45^0 = 8\sqrt{2} \text{ poundals}$$
$$ON = OB \sin 45^0 = 8\sqrt{2} \text{ poundals.}$$

Resolve OC into OL and OK. Then

$$OL = OC \cos 60^0 = \tfrac{1}{2} \times 10 \text{ poundals}$$
$$OK = OC \sin 60^0 = \tfrac{1}{2}\sqrt{3} \times 10 \text{ poundals.}$$



Resolve OD into OK and OT. Then

$$OK = OD \sin 120^0 = \tfrac{1}{2}\sqrt{3} \times 10 \text{ poundals}$$
$$OT = OD \cos 120^0 = -\tfrac{1}{2} \times 10 \text{ poundals.}$$

Hence $OA + OM + OL + OT = (8 + 8\sqrt{2} + 5 - 5)$ poundals
$= 8 + 8 \times 1{\cdot}41420\ldots$poundals $= 19{\cdot}314$ poundals.

And $ON + OK + OK = (8\sqrt{2} + 5\sqrt{3} + 5\sqrt{3})$ poundals
$= (8 \times 1{\cdot}41420 + 10 \times 1{\cdot}7320)$ poundals $= 28{\cdot}634\ldots$poundals,

$\therefore$ the resultant of the forces is

$$\sqrt{\{(19{\cdot}314)^2 + (28{\cdot}634)^2\}} \text{ poundals,}$$

that is $34{\cdot}5$ poundals nearly.

And the direction of the resultant makes an angle, whose tangent is $\frac{19314}{28634}$, that is, about $39^0\ 39'$ with OA.

EXAMPLES. XXIX. a.

1. Forces of 3 poundals and 4 poundals at right angles to each other act upon a mass of 5 lbs.; find the acceleration produced. How long will the mass take to move 18 ft. from rest?

2. Equal forces of 10 poundals at an angle of 120^0 to each other act on a mass of 1 lb.; how long will the mass take to acquire a velocity of 30 miles an hour?

3. Equal forces, each of the weight of 1 cwt., act on a ton placed on a smooth horizontal plane. The directions of the forces are respectively N and W. If at the instant the forces begin to act the ton is at rest, where will it be and what velocity will it have after 5 seconds?

4. Two horses, attached by ropes to a mass of 1 ton placed on a smooth horizontal plane, pull at right angles to each other, one with a force equal to the weight of 3 cwt. and the other with a force equal to the weight of 4 cwt.; find the direction of motion and the acceleration of the mass.

5. If in Question **4** the plane is rough so that there is a force of friction which retards the motion equal to $\frac{1}{10}$th of the pressure of the mass on the plane; find the acceleration.

6. Horizontal forces of 80 poundals, 160 poundals, 200 poundals act upon a mass of 1 ton placed on a smooth horizontal plane; the forces are represented by OA, OB, OC where $AOB=45^0$, $AOC=60^0$ and AC points westwards. If at the instant at which the forces begin to act the ton is at rest, where will it be at the end of 5 seconds?

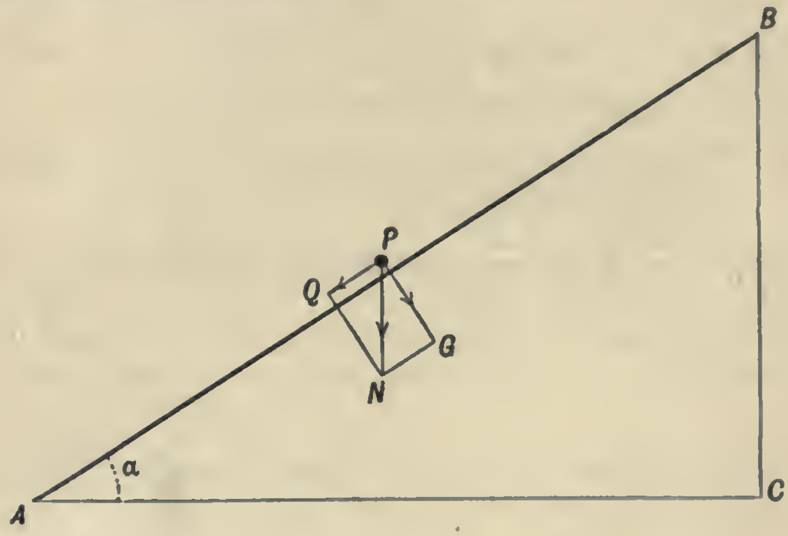
7. Forces of 2 poundals, 3 poundals and 1 poundal act along lines OA, OB, and OC so that the angle $AOB=45^0=BOC$ upon a mass of 4lbs.; find the acceleration produced.

8. Forces of 2 poundals, 3 poundals and 1 poundal act, along lines OA, OB and OC, so that the angle $AOB=30^0$, $BOC=15^0$, upon a mass of 1 lb.; find the acceleration produced.

9. Forces OA, OB, OC, OD act upon a mass of 100 lbs. such that $OA=10$ poundals, $OB=10\sqrt{2}$ poundals, $OC=20$ poundals, $OD=40$ poundals and $AOB=45^0$, $AOC=90^0$, $AOD=120^0$, find the acceleration produced.

10. Forces of 40 lbs. wt., 60 lbs. wt. and 80 lbs. weight act upon a mass placed on a horizontal plane. The forces are in one vertical plane and act at angles of 30^0, 60^0 and 120^0 to the horizontal plane. Find the magnitude of the mass that being under the action of gravity it shall not leave the plane.

Example ii. *A mass* m pounds is *placed under the action of gravity on a smooth inclined plane; find its acceleration.*



Let AB be the inclined plane; draw AC horizontal and BC vertical.

Let the angle BAC be α.

The weight of the mass P is a force mg poundals vertically downwards; let PN represent this force.

Resolve this force into its two components PQ and PG, in the directions along and perpendicular to PA respectively.

Then instead of mg poundals vertically downwards we have

$$PQ = mg \sin \alpha \text{ poundals along } PA$$

and

$$PG = mg \cos \alpha \text{ poundals along } PG.$$

Now the presence of the plane prevents any motion in the mass in the direction PG.

Also, the plane is '**smooth**'; by which is meant, that the stress between the mass and the plane must be perpendicular to the plane.

These conditions will both be satisfied if we suppose that the plane exerts a pressure on the mass, in the direction GP, just sufficient to prevent the mass from having any motion in that direction.

Hence we say, that the pressure of the plane on the mass in the direction GP is $mg \cos \alpha$ poundals.

Thus, the forces acting on the mass are $mg \sin \alpha$ poundals along PA, $mg \cos \alpha$ poundals along PG and $-mg \cos \alpha$ poundals in the same direction.

Therefore, the resultant force is $mg \sin \alpha$ poundals along PA.

This force produces in m pounds the acceleration in the direction PA

$g \sin \alpha$ *celos.*

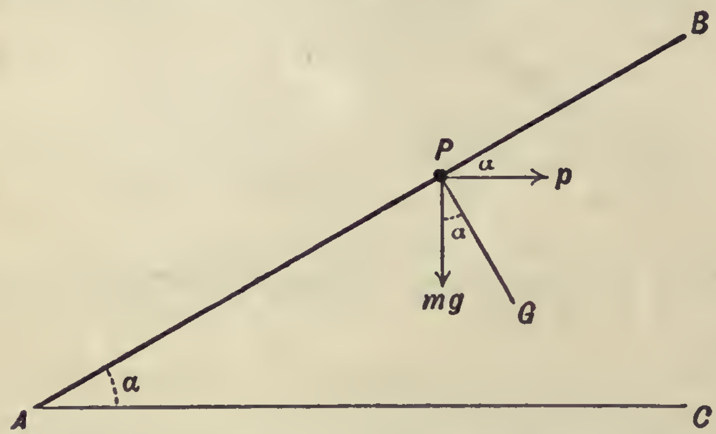
Example iii. *A mass* m lbs. *is placed on a smooth inclined plane and is acted on by its own weight and by a horizontal force* p poundals; *find* its acceleration.

With the construction of Example i. resolve the weight and the force p poundals along and perpendicular to the plane.

Case (i). We have if p poundals act as the forces in figure I.

$$(mg \sin \alpha - p \cos \alpha) \text{ poundals, along } PA,$$

$$(mg \cos \alpha + p \sin \alpha) \text{ poundals} - \text{the pressure of the plane, along } PG.$$



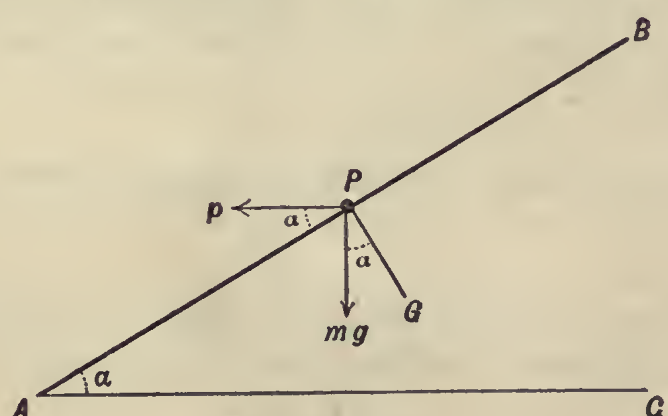
Assuming that the resultant acceleration is along AB, it follows that the resultant force must be along AB; so that

the pressure of the plane on the mass $= (mg \cos \alpha + p \sin \alpha)$ poundals; therefore the resultant force is $(mg \sin \alpha - p \cos \alpha)$ poundals along PA, which produces in mass m lbs. the acceleration

$$\left(g \sin \alpha - \frac{p \cos \alpha}{m}\right) \text{ celos along } PA.$$

If this acceleration happens to be zero the forces are in equilibrium; that is, if $mg \sin \alpha = p \cos \alpha$.

Case (ii) if p act as in figure II., then as before,



the pressure of the plane on the mass $=(mg\cos\alpha-p\sin\alpha)$ poundals.

And the acceleration along PA

$$\left(g\sin\alpha+\frac{p\cos\alpha}{m}\right)\text{ celos.}$$

EXAMPLES. XXIX. b.

1. A mass of 10 lbs. is placed on a smooth inclined plane of angle 30^0; how long will it take to slide 32 feet down the plane from rest?

2. A mass of 100 lbs. on a rough inclined plane is sliding down the plane whose angle is 45^0 with uniform velocity. What is the force of friction which is acting upon it?

3. A mass of 1 cwt. is sliding down a rough inclined plane whose angle is 30^0 and it goes 48 feet from rest in 8 seconds; what is the force of friction which is acting upon it?

4. A particle slides down a smooth inclined plane whose inclination to the horizon is 30^0; shew that it takes twice as long to descend a given vertical distance as a particle falling freely under the action of gravity.

5. A mass of 1 cwt. is placed on an inclined plane whose angle is 30^0 and the friction between the mass and the plane is known to be $\frac{1}{10}$ of the pressure of the mass on the plane when it is in motion; find the acceleration.

6. A train of 100 tons is on a *smooth* incline of 1 in 100 ($\sin\alpha=\frac{1}{100}$), what force must the engine exert in order to maintain a uniform velocity when ascending the incline?

7. A train of 100 tons is on an incline of 1 in 50 ($\sin\alpha=\frac{1}{50}$); find the retarding force of the breaks that they may be able to bring the train to rest in 220 yds. when going 30 miles an hour, (i) when ascending the incline, (ii) when descending the incline.

8. A mass of m lbs. is let fall from rest under the action of gravity and is acted on by a horizontal force of 24 poundals; find how long it will take to fall vertically a distance of 144 ft.; and how far it will move horizontally in the same time.

9. Prove that the mass in Example 8 moves in a straight line.

10. A mass P is drawn up a smooth plane inclined at an angle of 30^0 to the horizon by means of a mass Q which descends vertically, the masses being connected by a string which passes over a small pulley at the top of the plane; prove that when the masses are equal the acceleration is $\frac{1}{4}g$ celos.

11. Find the ratio of the masses of P and Q in Question 10 that they may have no acceleration.

12. Suppose that P in Question **10** starts from rest at the bottom of the plane and the interval t seconds occupied by Q in drawing P to the top of the plane is noticed; then P and Q being interchanged the interval t' seconds occupied by P in drawing Q up the plane is noticed; shew that if $t=2t'$ then $Q=\frac{2}{3}P$.

13. A mass m lbs. is drawn up a smooth plane inclined at an angle α to the horizon, by means of a mass m' lbs. which descends vertically, the masses being connected by a string which passes over a small pulley at the top of the plane; prove that the acceleration is

$$\frac{m'-m\sin\alpha}{m+m'}\,g \text{ celos.}$$

14. Two masses m lbs. and m' lbs. are connected by a light string which passes over a small smooth pulley fixed at the top of two smooth inclined planes of equal heights placed back to back, the inclination of the planes being each 30^0; shew that when the masses are equal they have no acceleration.

15. In Question **14** shew that when $m'=\frac{1}{2}m$ the acceleration is $\frac{1}{6}g$ celos.

16. If in Question **14** the inclinations of the planes to the horizon are α and α' respectively, prove that the acceleration is

$$\frac{m\sin\alpha-m'\sin\alpha'}{m+m'}\,g \text{ celos,}$$

and that the tension of the string is

$$\frac{mm'(\sin\alpha+\sin\alpha')}{m+m'}\,g \text{ poundals.}$$

17. Determine the acceleration due to gravity from the following experiment. A mass descending vertically under the action of gravity is observed to draw an equal mass a distance of 25 ft. from rest in $2\frac{1}{2}$ seconds up a smooth plane inclined at an angle 30^0 to the horizon, by means of a light string passing over a smooth small pulley at the top of the plane.

Example iv. *To find the* interval occupied *by a particle in sliding down a smooth inclined plane of length* s ft. *and inclination* α.

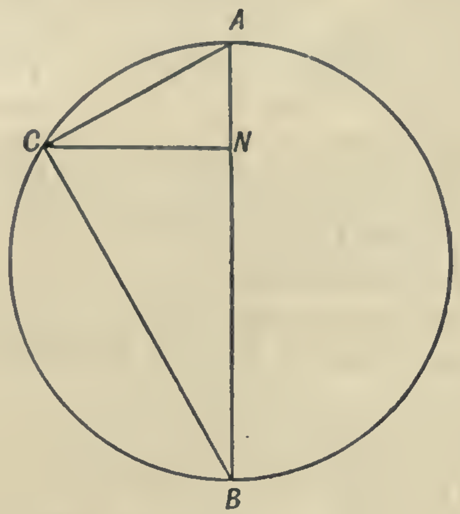
By Example i. the particle has an acceleration $g \sin \alpha$ celos down the plane.

The interval t seconds, occupied by a point in moving from rest over a distance of s ft. with acceleration $g \sin \alpha$ celos, is given by

$$s = \tfrac{1}{2}(g \sin \alpha)t^2,$$

whence,
$$t = \sqrt{\left\{\frac{2s}{g \sin \alpha}\right\}}.$$

Example v. *A series of* chords *are drawn from the highest point of a fixed vertical circle; prove that the intervals occupied by a particle in sliding from rest down each of these chords* (supposed smooth) *are all equal.*



Using the notation of Example iv, let ABC be the circle; AC one of the chords; CN horizontal; then $AC = s$ ft. and $ACN = \alpha$.

The interval occupied in sliding down AC, by Example iv, is

$$\sqrt{\left\{\frac{2s}{g \sin \alpha}\right\}} \text{ seconds.}$$

But
$$AC = AB \sin \alpha,$$

That is, $\dfrac{s}{\sin \alpha} = AB = 2r$; the length of the radius being r feet.

Hence, the interval in question is

$$2\sqrt{\frac{r}{g}} \text{ seconds,}$$

which is the same for each chord. Q. E. D.

EXAMPLES. XXX.

1. Prove that the interval occupied by a particle in sliding down an inclined plane of height h ft. and inclination α is

$$\frac{1}{\sin\alpha}\sqrt{\left\{\frac{2h}{g}\right\}}.$$

2. A train is on a smooth incline of 1 in 100; find how long it will take to go a mile from rest. [N.B. by an incline of 1 in 100 is meant that in the figure of page 109, the incline is such that $BC=1$ and $AB=100$.]

3. Find the velocity acquired at the end of the mile by the train in Example **2**. If another train start to go up a smooth incline of 1 in 50 with that velocity, shew that it will go half a mile up the plane.

4. A particle slides down a smooth straight tube of length l inclined at an angle α to the horizon, and then falls freely under the action of gravity; if the lower end of the tube is h feet above a horizontal plane, find when and where the particle will strike the plane.

5. A particle slides down a smooth straight tube, whose upper point is at a height h ft. from a horizontal plane, and then falls freely under the action of gravity; prove that it will in all cases reach the horizontal plane with $\sqrt{(2gh)}$ velos.

6. A particle is sliding down a smooth tube inclined at an angle to the horizon, and it takes twice as long to descend any vertical distance as a particle falling freely; find the inclination of the tube.

7. Find the direction in which a smooth tube must be drawn, so that one end being at a fixed point and the other end on a fixed straight line, a particle sliding down it may reach the straight line in the shortest possible time.

NOTE. The above problem is often stated thus. Find the line of **quickest descent** from a given point to a given straight line.

8. Prove that the time of sliding down any chord, supposed smooth, of a vertical circle drawn to the lowest point, is constant.

9. Find the line of quickest descent from a given straight line to a given point.

*SECTION III.

ILLUSTRATIONS.

CHAPTER IX.

PROJECTILES.

117. A particle which, having been projected from any point in any direction, is then supposed to move under the action of its own weight only, is called a **projectile.**

The resistance of the air is here neglected. This however has a very considerable effect on the motion of any mass actually projected, and renders this solution of the problem of little practical use.

118. When a projectile is projected vertically upwards its path is a vertical straight line.

This case has already been considered [Art. 46].

119. When a particle is projected in a direction which is not vertical, its path is a curved line.

This curved line is in one plane, which contains the vertical and the original direction of projection.

For there is no force tending to take it out of this plane.

120. We shall denote the velocity of projection by u velos, and shall suppose that its direction makes an angle α (radians) with the horizon.

121. The resolved parts of the velocity of projection are

(i) $u \cos \alpha$ (velos) in the horizontal direction,

(ii) $u \sin \alpha$ (velos) in the vertical projection.

122. PROP. *The* **horizontal velocity** *of a projectile is* constant.

By the *horizontal velocity* is meant *the resolved part in a horizontal direction* of the velocity at any instant.

The acceleration of the particle is always *vertically* downwards; so that after any interval, the *additional* velocity of the particle is also vertically downwards, and consequently has no resolved part in the horizontal direction.

The horizontal velocity is therefore unaltered. Q. E. D.

123. PROP. *The* **vertical velocity** *of a projectile after t seconds is* $(u \sin \alpha - gt)$ velos.

This is the expression of the fact that initially the particle has a vertical velocity $u \sin \alpha$ velos, and that it has the vertical acceleration $-g$ celos.

124. PROP. *The* **velocity** *of a projectile after t* seconds *is*
$$\sqrt{\{(u \cos \alpha)^2 + (u \sin \alpha - gt)^2\}} \text{ velos},$$
and if θ be the angle which its direction makes with the horizon
$$\tan \theta = \frac{u \sin \alpha - gt}{u \cos \alpha}.$$

For, the velocity of the projectile at any instant is the resultant of the horizontal and vertical velocities at that instant [See Art. 95].

125. PROP. *To find when and where a projectile is at its* **greatest** *vertical* **height**.

The vertical velocity is $(u \sin \alpha - gt)$ velos.

When this is positive the particle is going upwards, and when this is negative it is going downwards; when this is zero the projectile has reached its greatest height; that is, when
$$t = \frac{u \sin \alpha}{g}.$$

The vertical distance after t seconds is

$$(u \sin \alpha t - \tfrac{1}{2}gt^2) \text{ feet,}$$

in this putting $\dfrac{u \sin \alpha}{g}$ for t, we find for the **greatest height**

$$\frac{u^2 \sin^2 \alpha}{2g} \text{ feet.}$$

The horizontal distance at the same instant is found by putting $\dfrac{u \sin \alpha}{g}$ for t in $u \cos \alpha t$; it is

$$\frac{u^2 \cos \alpha \sin \alpha}{g} \text{ feet.}$$

EXAMPLES. XXXI.

1. A particle is projected under the action of gravity with 25 velos making an angle whose tangent is $\frac{3}{4}$ with the horizon; find when it is moving with 20 velos.

2. A particle is projected with 416 velos making an angle with the horizon whose sine is $\frac{12}{13}$; shew that its least velocity is 160 velos, and that when it has that velocity it is moving horizontally.

3. Find when the projectile in Example 2 has a velocity of 384 velos; find also its direction of motion at that instant.

4. Shew that the total velocity of a projectile projected with given initial velocity at a given angle is least when it is moving horizontally.

5. A projectile is projected with u velos making an angle α with the horizon; prove that it will be moving in a direction at right angles to the direction of projection after $\dfrac{u}{g}$ cosec α seconds.

6. Two particles are projected simultaneously in the same vertical plane with velocities u velos and v velos at angles α and β to the horizon respectively; shew that their velocities will be parallel after

$$\frac{uv \sin(\alpha - \beta)}{g(v \cos \beta - u \cos \alpha)} \text{ seconds.}$$

7. A particle projected with u velos at an angle α, is moving in a direction making an angle β with the horizontal plane after

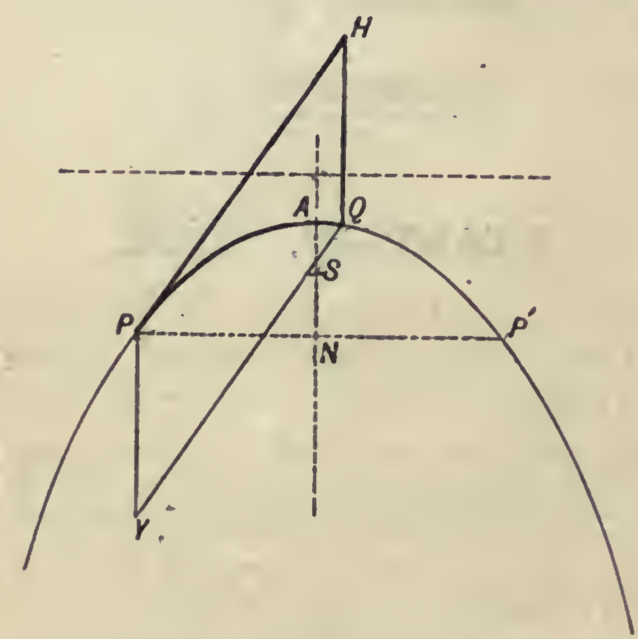
$$\frac{1}{g}(u \sin \alpha - u \cos \alpha \tan \beta) \text{ seconds.}$$

126. PROP. *The* path *of a projectile is a* **Parabola.**

Let the initial velocity be u velos making an angle α with the horizon.

Let P be the point of projection; Q the position of the projectile after t seconds. Draw PP' horizontal; draw PV vertically downwards; draw PH making an angle α with PP'; draw QH vertical and QV parallel to HP.

Then [Art. 91] $PH = ut$ ft.; $HQ = \frac{1}{2}gt^2$ ft.



Therefore
$$QV^2 = PH^2 = u^2t^2$$
$$= u^2 \times \frac{2HQ}{g} = \frac{2u^2}{g} \times PV$$
$$= 4SP \,.\, PV; \quad \text{where } 4SP = \frac{2u^2}{g} \text{ ft.}$$

But in the Parabola, $QV^2 = 4SP \,.\, PV$; where SP is the distance of the focus from P.

Therefore, if we draw a parabola, touching PH at P, whose axis is vertical and vertex upwards, such that
$$4SP = \frac{2u^2}{g} \text{ ft.},$$
then Q will always be on this parabola. Q. E. D.

127. PROP. *The* **speed** *of a projectile at any point of its path is equal to that due to a fall from the directrix.*

Let S be the focus, let P be the point of projection; then, SP is the vertical distance of P from the directrix; and, by the last Article, $SP = \frac{u^2}{2g}$ ft.

Let h be the number of ft. in SP; then, the velocity v due to falling with acceleration g, a distance h ft. is given by

$$v^2 = 2gh$$

but from above $\qquad u^2 = 2gh;$

whence $\qquad u = v.$

Thus the speed of projection is equal to that due to falling from the directrix.

And since this is true at the point of projection, it must also be true at every point of the path; for the particle might be '*projected*' from any point in its path.

NOTE. This proposition may be stated as follows; When any particle has fallen vertically under the action of gravity from a point on the directrix to a point on the parabola, it has a vertical velocity downwards, whose *magnitude* is equal to the velocity which the particle describing the parabola has when at that point. It must be noticed that these two velocities are not in the same direction.

It follows that the horizontal speed is that due to falling from the directrix to the vertex, and hence that the number of feet in the latus-rectum of the parabola is $\frac{2u^2 \cos^2 \alpha}{g}$.

*128. Another proof of Art. 127 is as follows.

Let the tangent at P cut the axis of the parabola in T. The vertical velocity at A is zero; so that the vertical velocity at $P\,(u \sin \alpha)$ is that due to falling a vertical distance AN; hence $u^2 \sin^2 \alpha = 2g\,AN$.

[Now in the parabola with the usual notation

$$2AN = NT = PT \sin \alpha = 2PY \sin \alpha = 2SP \sin^2 \alpha],$$

whence $\qquad u^2 \sin^2 \alpha = 2g\,SP \sin^2 \alpha,$

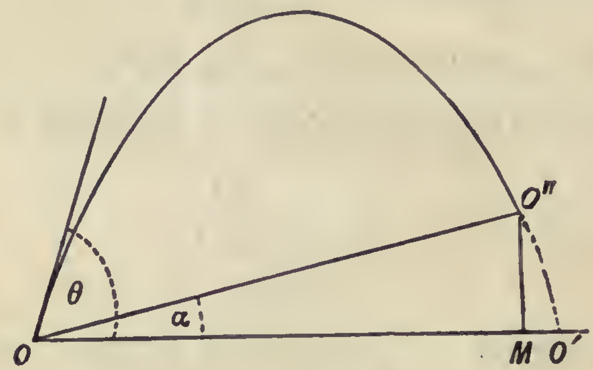
or $\qquad u^2 = 2g\,SP. \qquad$ Q. E. D.

Example i. *To find the* **range** *and* **time of flight** *on an* **inclined plane,** angle α, *which passes through the point of projection.*

Resolve the velocity of projection along and perpendicular to the inclined plane; the resolved parts are, in velos,

initially, $u\cos(\theta-\alpha)$; $u\sin(\theta-\alpha)$;

after t secs, $u\cos(\theta-\alpha)-(g\sin\alpha)t$; $u\sin(\theta-\alpha)-(g\cos\alpha)t$.



The resolved parts of the distance in the same directions are in ft., after t secs., $u\cos(\theta-\alpha)t-\frac{1}{2}(g\sin\alpha)t^2$; $u\sin(\theta-\alpha)t-\frac{1}{2}(g\cos\alpha)t^2$.

The particle strikes the plane again when the distance perpendicular to the plane, viz. $u\sin(\theta-\alpha)t-\frac{1}{2}(g\cos\alpha)t^2$, is zero;

that is, when $$t=\frac{2u\sin(\theta-\alpha)}{g\cos\alpha}.$$

Let OO'' in the figure be the range; draw $O''M$ perpendicular to the horizontal line OO'; then OM = horizontal velocity × time of flight

$$=u\cos\theta\times\frac{2u\sin(\theta-\alpha)}{g\cos\alpha},$$

also, $$OO''=\frac{OM}{\cos\alpha},$$

that is, the range is $$\frac{2u^2}{g}\frac{\sin(\theta-\alpha)\cos\theta}{\cos^2\alpha}\text{ feet.}$$

Example ii. *Prove that in Example* i. *the particle at the middle of the time of flight is* (i) *vertically above the middle point of the range and* (ii) *is moving parallel to the inclined plane.*

The first statement is most easily proved by observing that if the projectile carries a fine thread which hangs always vertical, this thread travels with uniform velocity in a horizontal direction; and that therefore the point in which this vertical thread cuts the inclined plane travels with *uniform velocity along the plane.*

To prove the second statement, we observe that the velocity of the projectile *perpendicular* to the inclined plane is zero, when

$$u \sin(\theta - \alpha) - g \cos \alpha \times t \text{ is zero,}$$

that is, when $$t = \frac{u \sin(\theta - \alpha)}{g \cos \alpha},$$

which is at the middle of the time of flight.

Therefore at the middle instant of the time of flight the velocity of the projectile is parallel to the plane.

EXAMPLES. XXXII.

1. A train is moving at the rate of 45 miles an hour, when a ball is dropped from the roof inside one of the carriages; prove that the ball describes a parabola in space, and find the position of the axis and directrix.

2. A boy throws a stone from the top of a cliff 128 ft. high with 64 velos at an angle 30^0 to the horizon; find how far from the foot of the cliff it will strike the sea and with what velocity.

3. Prove that, neglecting the resistance of the air, the range of a projectile on a horizontal plane is greatest when the angle of projection is 45^0.

4. Find with what velocity a man must be able to throw that he may just be able to throw a cricket ball 100 yards.

5. A man throws a ball with a velocity 50 velos; in what direction must he throw it that he may just strike the top of a pole 40 ft. high at a distance of 100 ft. from him?

6. A particle is projected with velocity u velos at an elevation α; find its least velocity, and find when and where it attains that velocity.

7. A particle is describing a parabola under the action of gravity, and at the instant when it has its least velocity that velocity is doubled; shew that the time of reaching the horizontal plane from which it was projected is unaltered.

8. From the top of a tower I project particles in a horizontal direction; prove that the distances, from the foot of the tower at which the particles strike the horizontal plane on which it stands, are proportional to the velocities of projection.

9. A projectile has initially u velos and its angle of projection is α; prove that its distance from the point of projection after t seconds is

$$\sqrt{(u^2t^2 - u \sin \alpha \, gt^3 + \tfrac{1}{4} g^2 t^4)} \text{ feet.}$$

10. Find the angle of elevation that the horizontal range may be equal to the distance of the point of projection from the directrix.

11. Shew that the greatest range on an inclined plane of 30^0, is two-thirds of the greatest range on a horizontal plane, with the same initial velocity.

12. A straight smooth tube AB, 72 ft. long, is placed at an angle 30^0 to the horizon so that its lower end B is at a height 16 ft. from a horizontal plane; a particle is allowed to slide from A through the tube and then to describe a parabola freely; find how long after leaving the tube the particle strikes the ground.

13. In Question **12**, find how far from the point vertically under B is the point at which the particle strikes the ground.

14. A straight smooth tube AB, a ft. long, inclined at an angle α to the horizon is fixed so that the lower end B is h ft. from a horizontal plane. A particle slides from A through the tube and then describes a parabola freely, striking the plane at C; find the time of falling from B to C and the distance BC.

15. A man in a railway carriage, moving with uniform velocity 30 miles per hour, throws a ball so that it goes up four feet and returns to the point of projection; shew that the ball describes a parabola relatively to the earth and find how far the train moves while it does so.

16. Two men 4 ft. 7 in. apart in a railway carriage which is moving uniformly with a velocity of 45 miles an hour throw a ball from one to the other, projecting it so that its time of flight is $\frac{1}{8}$ second. Shew that the ball describes a parabola in space; and taking its path in the carriage to be perpendicular to the rails, find its range in space.

17. The top of a railway carriage is 12 ft. from the ground, and it is moving at the rate of $7\frac{1}{2}$ miles per hour; a man jumps from the top, giving himself a horizontal velocity perpendicular to the rails of $8\frac{1}{4}$ velos; find the velocity with which he will reach the ground.

18. A particle is projected with velocity u at an angle α; with what velocity must I move on a horizontal plane that I may be always vertically under the projectile?

19. A particle is projected from a point on an inclined plane of angle β with velocity u at an angle α to the horizon so that its path is in the same plane as the line of greatest slope in the plane; with what velocity must I move on the plane so as to be always vertically under the particle?

20. **Morin's machine** consists of a vertical circular cylinder which is made to rotate with uniform velocity, while one of the weights of an Atwood's machine having a pencil fixed to it descends with constant acceleration in a vertical line, and as it does so marks a line on the surface of the cylinder; shew that if the cylinder is covered with a sheet of paper, which after the pencil has made its mark upon it, is taken off and flattened, then the trace of the pencil on the paper will be a parabola.

21. Find the length of the latus rectum of the parabola described on a Morin's machine, in which the circumference of the cylinder is 3 ft. and the weights in the Atwood's machine are $15\frac{1}{2}$ oz. and $16\frac{1}{2}$ oz. respectively $[g=32]$, the cylinder making a complete revolution in 6 seconds.

22. A particle is projected with velocity u velos at an angle α; shew that it reaches the further extremity of the latus rectum in

$$\frac{1}{g}(u \sin \alpha + u \cos \alpha) \text{ seconds.}$$

23. A particle slides down a smooth inclined plane of angle α, having initially a horizontal velocity on the plane (in a direction perpendicular to the line of greatest slope) of u velos; shew that the particle traces out a parabola on the plane and find its latus rectum.

24. A particle is projected with velocity u velos at an angle α; shew that the focus is $\dfrac{u^2}{2g}(\sin^2\alpha - \cos^2\alpha)$ above the horizontal plane through the point of projection.

25. The focus of the path of a projectile is above or below the horizontal plane through the point of projection according as the vertical component of the velocity of projection is greater or less than the horizontal component.

26. A particle slides down a smooth straight tube and then falls freely under the action of gravity; prove that the directrix of its parabolic path passes through the upper end of the tube.

CHAPTER X.

OBLIQUE IMPACT.

129. DEF. An impact which is not *direct*, as defined in Art. 65, is said to be **oblique.**

130. DEF. When two particles impinge obliquely, the line of action of the stress between the particles set up by the impact is called the **line of impact.**

131. PROP. *To find the* velocities after impact *of two masses of given elasticity which impinge obliquely with given velocities.*

Resolve the velocity before impact of each particle into two *resolved parts*, one along, the other perpendicular to, the line of impact;

Then, since the stress is in the line of impact, the velocity which it produces in either mass has no resolved part in the direction perpendicular to this line of impact.

Therefore the resolved parts, of the velocities of the masses after impact in the direction *perpendicular* to the line of impact, are the *same* as before impact.

The stress produces its effect on the resolved parts of the velocities in the line of impact exactly as if the other resolved parts did not exist.

So that we find the new velocities in the line of impact by the method of Art. 72.

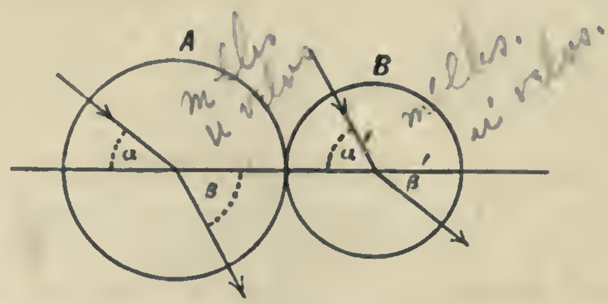
Each of these new velocities is then compounded with the resolved part perpendicular to the line of impact (which is the same as that before impact).

The resultant velocities thus obtained are the required velocities after impact of the particles respectively.

Example. *Two* smooth spheres A *and* B, *of* m lbs. *and* m′ lbs. *having velocities* u velos *and* u′ velos, *making* angles α *and* α' *respectively with the line of impact, impinge on one another; find their* velocities *after impact.*

The resolved parts of the velocities perpendicular to the line of impact are, in velos, $u \sin \alpha$, $u' \sin \alpha'$, respectively.

These are [Art. 131] also the resolved parts in their own direction of the required new velocities.



The resolved parts of the velocities along the line of impact are

$$u \cos \alpha, \quad u' \cos \alpha', \text{ respectively.}$$

Let the elasticity of the spheres be e. Then, since the problem is now one of direct impact, we have, by Arts. 68 and 72, if v, v' be the velocities after impact in the line of impact

$$mv + m'v' = mu \cos \alpha + m'u' \cos \alpha',$$

and,
$$v - v' = -e\,(u \cos \alpha - u' \cos \alpha')$$

whence,
$$v = \frac{(m - em')\, u \cos \alpha + m'\,(1 + e)\, u' \cos \alpha'}{m + m'},$$

$$v' = \frac{(m' - em)\, u' \cos \alpha' + m\,(1 + e)\, u \cos \alpha}{m + m'}.$$

The resultant velocity of A is therefore $\sqrt{\{u^2 \sin^2 \alpha + v^2\}}$ in the direction making the angle whose tangent is $\frac{u \sin \alpha}{v}$ with the line of impact; a similar statement gives the velocity of B.

EXAMPLES. XXXIII.

1. A sphere A of elasticity e impinges with $20\sqrt{2}$ velos on an equal sphere B at rest, the line of impact making an angle of 45^0 with the direction of motion of A; find the velocity of B after impact.

2. A sphere impinges on an equal sphere at rest, find the condition that after impact their velocities may be at right angles.

3. A sphere A impinges on a sphere B of equal mass; their velocities before impact are at right angles and equally inclined to the line of impact and are equal in magnitude; shew that when $e=\frac{1}{3}\sqrt{3}$ their velocities after impact are inclined at an angle 60^0.

4. A glass ball of elasticity e when moving horizontally with velocity u velos, is struck by an equal ball moving vertically with u' velos, so that the line of impact is vertical; find their velocities after impact.

5. A bird weighing 1 lb. moving horizontally with 85 velos is struck by a bullet weighing 1 oz. moving vertically upwards with a velocity of 1020 velos; find the subsequent velocity of the bird supposing the bullet to lodge in the bird; and supposing it killed by the shot when at a height of 100 ft., find when it will fall to the ground, neglecting the resistance of the air.

6. A bird weighing 2 lbs. moving horizontally with 30 velos is struck by a bullet of 1 oz. moving horizontally, but at right angles to the path of the bird, with a velocity 1320 velos; find the velocity after impact supposing the bullet to lodge in the bird; and supposing the bird killed when at a height of 128 ft., find how long it will be before it strikes the ground.

132. When a smooth elastic ball impinges obliquely on a plane fixed to the ground, the problem is treated as in Art. 73.

Examples. A sphere *of* elasticity e *is projected with* velocity u *at an* angle α to the horizon *from a point in a smooth horizontal plane, against which it impinges and rebounds; investigate the motion.*

The resolved parts of the velocity of projection are (in velos)

$$u \sin \alpha, \text{ vertically}; \; u \cos \alpha, \text{ horizontally}.$$

The particle first describes a parabola.

On reaching the horizontal plane again the particle has velocity

$$-u \sin \alpha, \text{ vertically}; \; u \cos \alpha \text{ horizontally}.$$

The impact does not alter the horizontal velocity; the vertical velocity after impact is $eu \sin \alpha$.

The particle now describes another parabola the velocity of projection being $u \cos \alpha$ velos horizontally, $eu \sin \alpha$ velos vertically. And so on.

Thus, after the nth impact the vertical velocity is

$$e^n u \sin \alpha; \text{ the horizontal velocity is } u \cos \alpha.$$

The *range* of any rebound is (in feet)

$$\frac{2\,(\text{vertical velocity} \times \text{horizontal velocity})}{g}.$$

The distance from the point of projection at which the sphere will strike the plane after the nth rebound is

$$\frac{2u^2 \sin\alpha \cos\alpha}{g}\{1+e+e^2+\ldots+e^n\};$$

the limit of which, when n is increased without limit, is

$$\frac{2u^2 \sin\alpha \cos\alpha}{g(1-e)}.$$

The meaning of this is, that after reaching this distance from the point of projection, the sphere will cease to rebound and will slide with velocity $u \cos\alpha$.

133. The following proposition is important.

PROP. *When two* masses *of* m lbs. *and* m′ lbs. *respectively of* elasticity e *impinge directly, if* v velos *and* v′ velos *be their velocities after impact, then*

$$mv^2 + m'v'^2 \textit{ is less than } mu^2 + m'u'^2.$$

We have, $mv + m'v' = mu + m'u'$

and, $v - v' = -e(u-u')$.

Therefore $(mv + m'v')^2 = (mu + m'u')^2$

and, $mm'(v-v')^2 = mm'e^2(u-u')^2$.

Whence, by addition

$$(m+m')(mv^2+m'v'^2) = (m+m')(mu^2+m'u'^2) - mm'(1-e^2)(u-u')^2,$$

or, $$mv^2 + m'v'^2 = mu^2 + m'u'^2 - \frac{m\,m'}{m+m'}(1-e^2)(u-u')^2.$$

Now e is never greater than 1; so that $1-e^2$ cannot be negative; hence $mv^2 + m'v'^2$ is always less than $mu^2 + m'u^2$; except when $e = 1$, and in that case the two expressions are equal.

This result may be expressed (see Chapter XV.) thus; *The* kinetic energy *of two masses is diminished by direct impact, except when the* elasticity *of the masses is perfect.*

We leave as an exercise for the student the proof of the proposition that if v velos and v' velos be the velocities after oblique impact of masses m lbs. and m' lbs. then $mv^2 + m'v'^2$ is less than $mu^2 + m'u'^2$. See Ex. XXXIV. 17.

EXAMPLES. XXXIV.

1. A glass ball of elasticity $\frac{2}{3}$ is projected with 40 velos at an angle of projection whose sine is $\frac{4}{5}$, from a point on a horizontal pavement; find its range after one rebound.

2. A glass ball of elasticity $\frac{1}{2}$ is projected at an angle whose sine is $\frac{12}{13}$ with 156 velos from a point in a horizontal plane; find how far it will go before it ceases to rebound.

3. A particle of elasticity $\frac{3}{5}$ is projected with velocity 100 velos at an angle 60^0 to the horizon from a point in an inclined plane making an angle 30^0 with the horizon; find the velocity of the particle after one rebound from the plane.

4. A particle of elasticity e is projected with u velos at an angle α to the horizon and after striking a fixed vertical wall at a horizontal distance h ft. returns to the point of projection; prove that

$$hg(1+e)=2u^2 e \sin\alpha\cos\alpha.$$

5. Two particles of elasticity e are projected in exactly opposite directions from a point between two fixed vertical parallel planes and distant a ft. and b ft. from them respectively; shew that they will not meet again unless the directions of projection are perpendicular to the planes.

6. A particle of elasticity e is projected from a point half-way between two fixed parallel vertical walls $2a$ ft. apart, in a given direction, and after two rebounds returns to the point of projection; find the velocity of projection.

NOTE. When a sphere strikes a plane the angle its velocity makes with the normal to the plane is called the **angle of incidence**; the angle which its velocity after impact makes with this normal is called the **angle of reflexion**.

7. A sphere of elasticity $\frac{1}{4}$ impinges on a plane; find the angle of incidence that its direction after impact may be at right angles with its direction before impact.

8. Prove that when a perfectly elastic sphere impinges on a smooth plane the angles of incidence and reflexion are equal.

9. A perfectly elastic ball is projected on a smooth rectangular billiard table in a direction parallel to one of its diagonals; find the condition that after impinging on each of the four sides the ball will return to the point of projection.

10. If α be the angle of incidence and β the angle of reflexion of a sphere of elasticity e on a smooth plane, prove that $\cot\beta=e\cot\alpha$.

11. A sphere sliding on a smooth horizontal plane impinges in succession on two smooth vertical planes at right angles to each other; prove that the velocity of the sphere after the second impact is parallel to its velocity before the first impact.

12. Two spheres moving in parallel directions with equal and opposite mass-velocities impinge; prove that after impact they will move in parallel directions with velocities in the inverse proportion of their masses.

13. A smooth small sphere of elasticity e slides down a smooth inclined plane of height h and inclination α, and impinging on a smooth horizontal plane at the foot of the first describes a parabola; find the range.

14. A particle of elasticity e is projected from a horizontal plane so as to impinge on a given vertical wall at a distance a ft., and after one rebound on the horizontal plane returns to the point of projection; the velocity of projection being u velos, find the direction of projection.

15. A sphere of elasticity e is projected at an angle α from the middle of the floor of a rectangular room h ft. high, $2b$ ft. wide, so that after impinging on a wall, on the ceiling and on the opposite wall, it returns to the point of projection; find equations to determine the velocity of projection.

16. A smooth inelastic particle slides along the sides of a regular polygon of n sides under the action of no forces; prove that its velocity after passing m angular points is diminished in the ratio of $\left(\cos\frac{2\pi}{n}\right)^m$ to 1.

17. Two masses m lbs. and m' lbs., moving with u velos and u' velos respectively impinge, so that u, u' make angles α, α' with the line of impact; prove that if after impact they are moving with v velos and v' velos respectively, then

$$mv^2+m'v'^2=mu^2+m'u'^2-\frac{mm'}{m+m'}(1-e^2)(u\cos\alpha-u'\cos\alpha')^2.$$

CHAPTER XI.

NEWTON'S LAWS OF MOTION.

134. Newton's Laws of Motion are

LEX I. *Corpus omne perseverare in statu quo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.*

LEX II. *Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*

LEX III. *Actioni contrariam semper et æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*

They may be translated as follows:

I. *Every body perseveres in its state of rest or of moving uniformly in a straight line, except in so far as it is made to change that state by external forces.*

II. *The change of the mass-velocity of a body is numerically equal to the impulse which produces it and is in the same direction.*

III. *To every action there corresponds an equal and opposite reaction; that is to say, the actions of two bodies upon each other are always equal and in opposite directions.*

Newton's Laws, I and II, follow immediately from Art. 31, provided we define *Impulse* as the equivalent of the continuous action of a Force for some definite interval.

For, by Art. 31, a *force* is numerically equal to the mass-*acceleration* which it produces; therefore a force × an interval, that is, an *impulse* is numerically equal to the mass-*velocity* which it produces.

We may therefore in Art. 33 replace the words *force* and *acceleration* by the words *impulse* and *velocity*; and then Art. 33 is a statement of what is implied by Laws I and II.

Conversely, the statement of Art. 31 follows immediately from Laws I and II, provided we define *force* as, that whose continuous action for a definite interval is equivalent to an impulse; for by Law I, an *impulse* is numerically equal to the mass-*velocity* which it produces; therefore the ratio of the impulse to its interval, that is, its *force*†, must be numerically equal to the mass-*acceleration* which it produces.

The following is an illustration of the relation between Force and Impulse.

Consider a mass of 10 tons on a smooth horizontal plane (for instance a railway carriage on smooth horizontal rails), to which an horizontal impulse is applied by means of a hammer.

Suppose the mass of the hammer to be 2 lbs. and the velocity 32 velos; and suppose the hammer just brought to rest by the impact.

The impulse communicates the mass velocity 64 pound-velos to the 10 tons.

When the 10 tons is in motion, by making the hammer impinge again with a velocity 32 velos greater than the velocity of the 10 tons and bringing it relatively to rest by the impact, we can arrange that the impulse 64 pound-velos can be repeated any number of times.

Now suppose men employed to apply to the 10 tons in succession a series of impulses at intervals of $\frac{1}{100}$ of a second; *each* impulse communicating 64 pound-velos to the 10 tons, and suppose that they apply 100 impulses per second.

In such a case, if the force of each impulse lasts exactly $\frac{1}{100}$ of a second, and is uniform, then the 10 tons is acted on by a *continuous force*.

64 pound-velos are communicated to the 10 tons every $\frac{1}{100}$ of a second; that is, 6400 pound-velos per second, and

The motion is that produced by a force of 6400 poundals; or by the weight of about 200 lbs.

† In the above the force during the interval is considered as constant. When the force is not constant the interval must be taken so small that the force may be considered constant.

135. Let us suppose that there can be such a thing as an *instantaneous* impulse. Suppose an impulse ω pulses applied to a particle of unit mass (in the direction of the motion of the particle) at the beginning of each of a series of n intervals, each of τ seconds, such that $n\tau = t$.

The particle will receive at the beginning of each interval an additional velocity ω velos.

Next suppose the intervals diminished, their number increased, and the *ratio of the impulse to the interval* kept the same so that $\frac{\omega}{\tau}$ is constant; and let a stand for the ratio $\frac{\omega}{\tau}$.

Then in the limit, when the number n is infinitely increased, the motion of the particle is that of the uniformly increasing velocity of a celos.

136. Here, an impulse (which is equivalent to the continuous action of a force for some given interval) is considered as composed of series of infinitely small impulses applied at infinitely small intervals, the sum of these little intervals making the given interval.

The *magnitude* of a continuous force is measured by the *rate* at which it produces impulse.

137. It is not necessary however in the above to suppose the existence of an *instantaneous* impulse.

Let each impulse produce the velocity ω velos gradually and uniformly in the interval τ secs.; then, whether the intervals τ secs. are large or small, the motion of the particle is uniformly accelerated motion such as, by Art. 31, is produced by continuous force.

138. But just as the properties of curves are studied by supposing them to be derived from polygons, in which the sides are made to become infinitely numerous and infinitely short; so it is sometimes convenient to imagine our continuous force acting for a certain interval to be equi-

valent to a series of small impulses; these impulses being *instantaneous*, and becoming, in the limit, infinitely quick and infinitely small.

139. PROP. *To prove* the formula $s = ut + \frac{1}{2}at^2$ *of* Art. 21, *by* Newton's *method.*

Suppose a point to be moving in a straight line so that its velocity receives at the beginning of each one of a series of equal intervals of τ secs. an additional velocity ω velos.

Let n of these intervals of τ secs. make up t secs.

Let u velos be the velocity of the point at the beginning of the interval t seconds.

The velocity during the rth interval τ seconds is

$$(u + r\omega) \text{ velos.}$$

The distance passed over in the course of that interval is

$$(u + r\omega)\,\tau \text{ ft.}$$

So that the whole distance passed over in t seconds is

$$\{(u + \omega)\,\tau + (u + 2\omega)\,\tau + \ldots + (u + n\omega)\,\tau\} \text{ feet,}$$

that is $$\tfrac{1}{2}n\,\{2u\tau + (n + 1)\omega\tau\} \text{ feet,}$$

that is $$\{ut + \tfrac{1}{2}n\omega t + \tfrac{1}{2}\omega t\} \text{ feet. [Compare Art. 24.]}$$

Now suppose the interval τ seconds be diminished and consequently the number n increased so that $n\tau$ still equals t.

Suppose also that the *rate* at which velocity is added is kept unaltered, so that $\frac{\omega}{\tau}$ is a constant number $= \alpha$; and

$$n\omega = \frac{n\tau\omega}{\tau} = \frac{t\omega}{\tau} = \alpha t.$$

Then the distance passed over in t seconds is

$$\left\{ut + \tfrac{1}{2}\alpha t^2 + \tfrac{1}{2}\,\frac{\alpha t^2}{n}\right\} \text{ feet.}$$

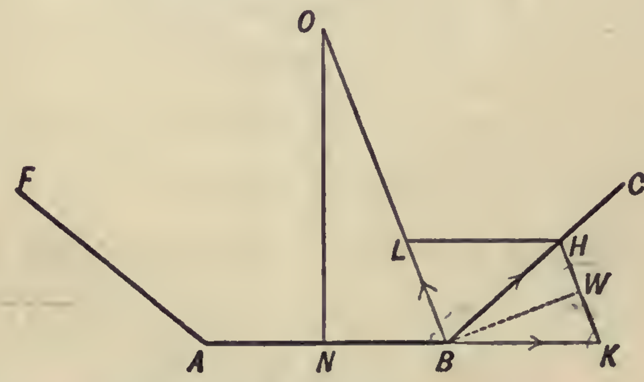
This is true however great n may be; and hence it is true when n is greater than any assigned number; in which case the distance passed over in t seconds is $(ut + \frac{1}{2}\alpha t^2)$ ft.

140. The following proposition is an instructive example of Newton's method.

141. PROP. *A point moves with constant speed* v velos *along the perimeter of a regular polygon* ABC... *of* n *sides inscribed in a circle of radius* r ft.; *prove that at each angle the point receives an additional velocity towards the centre of the circle; and that the* rate of additional velocity per second *is* $\frac{v^2}{r}$.

Let O be the centre of the circle.

Let $AB = s$ ft.; let τ seconds be the interval occupied in going from A to B; then, $s = v\tau$.



Draw ON bisecting AB at right angles.

Produce AB to K so that BK represents the velocity v velos along AB;

let BH represent the velocity v velos along BC; then

$$BH = BK.$$

Complete the parallelogram $BKHL$.

Draw BW bisecting HK at right angles.

Then, since $BK = BH$, the angle $BKH = BHK = HBL$.

But the angle $ABL = BKH$;

therefore the angle $ABL = HBL$.

Therefore BL produced passes through the centre O.

Now the velocity represented by BH is the resultant of the velocities BK, BL. That is, *the velocity* BL *when added to the velocity* BK *changes it into the velocity* BH.

In other words, when a point having v velos along AB changes its velocity to v velos along BC, it receives an additional velocity ω velos represented by BL, whose direction is towards the centre of the circle.

The rate of additional velocity per second is $\dfrac{\omega}{\tau}$.

But, $$\frac{\omega}{\tau} = \frac{\omega v}{s} = \frac{v^2}{8} \times \frac{\omega}{v}$$

and, $$\frac{\omega}{v} = \frac{HK}{BK} = \frac{2KW}{BK} = \frac{2NB}{OB} = \frac{s}{r},$$

Therefore, $$\frac{\omega}{\tau} = \frac{v^2}{s} \times \frac{\omega}{v} = \frac{v^2}{s} \times \frac{s}{r} = \frac{v^2}{r}.$$

That is, the required rate is $\dfrac{v^2}{r}$. Q. E. D.

142. PROP. *A point moves with the constant speed* v velos *along the circumference of a circle of* radius r; *prove that this point has constant acceleration* v^2/r celos *towards the centre of the circle.*

We consider a circle to be the curve which a regular polygon approaches as its limit, when the number of its sides is infinitely increased.

When the polygon in Art. 141 becomes a circle the additional velocity at each angular point is infinitely diminished, the number of angular points being infinitely increased.

Hence, instead of a series of sudden increases of velocity we have, for the circle, a *continuous* growth of velocity, always in the direction of the centre, at the rate v^2/r velos per second; that is, at the rate v^2/r.

N.B. Neither in the polygon nor in the circle does this increase of velocity alter the *speed*. It alters the velocity as regards *direction* only.

143. PROP. *A particle of mass* m lbs. *is describing a circle of radius* r ft. *with* v ~~velos~~; *prove that it is acted on by a force* $\frac{mv^2}{r}$ poundals *towards the centre.*

The particle has a constant acceleration $\frac{v^2}{r}$ celos towards the centre.

Acceleration never exists unless force causes it; hence there is a force $\frac{mv^2}{r}$ poundals towards the centre.

144. A force which always tends towards a fixed point is called a **centripetal** force.

Example. *A particle of* 1 lb., *fastened to a fixed point* O *by a light string* 2 ft. long, *describes the circle about* O *as centre once every second; find the* tension *of the string.*

[NOTE. In a question like this, gravity is not supposed to be acting. The particle might be moving on a perfectly smooth horizontal plane. The student should notice that unless some word such as, *weight*, *horizontal*, *vertical*, *heavy*, appears in a question, gravity is supposed not to act.]

The particle describes the circumference of a circle of 2 ft. radius in 1 sec.; therefore it moves at the rate of 4π ft. per sec.

Therefore by Art. 142 its acceleration is $\frac{16\pi^2}{2}$ celos.

Therefore it is acted on by $8\pi^2$ poundals.

The tension of the string is therefore equal to the weight of about

$$\frac{8 \times 22 \times 22}{32 \times 49} \text{ pounds, nearly;}$$

that is, to the weight of 2·47 lbs. nearly.

NOTE. When a mass is describing a circle with uniform speed, its velocity is being changed (not in magnitude, but) in direction. By reason of its inertia this change cannot be made in the velocity of the mass by anything except by external force. We have shewn that in order to effect this change a force of a certain magnitude must continuously act on the particle *towards the centre* of the circle. The inertia of the mass continuously opposes this force; and used to be said to be *centrifugal*. This word *centrifugal* is very misleading; for the student should carefully notice that if the *centripetal* force ceases, the motion of the mass is along a *tangent:* therefore the tendency of the mass should be called **tangential**, not centrifugal.

EXAMPLES. XXXV.

In the first four of the following questions gravity is not acting.

1. A point is describing a circle of radius 5 ft. with uniform speed 10 velos, find its acceleration.

2. A mass of 4 lbs. is describing a circle of radius 5 ft. with uniform speed of 10 velos; what force is acting upon it?

3. A mass m lbs. fastened by a string to a fixed point is describing a circle uniformly with v velos; the tension of the string is p poundals, find the length of the string.

4. A mass m lbs. is fastened by a string of length r feet to a fixed point O, and describes a circle about O n times per second; shew that the tension of the string is $4mn^2\pi^2r$ poundals.

5. A railway carriage is describing a curve on horizontal rails; shew that if it goes fast enough it must fall over.

6. A railway carriage of 10 tons is describing an arc of a circle of $\frac{1}{4}$ mile radius with a speed of 30 miles an hour; find the horizontal force which must be acting on the carriage.

7. A mass of 1 cwt. is in a railway truck which is moving, on horizontal rails on a curve whose radius is 400 yards, at the rate of 60 miles an hour; what is the horizontal stress between 1 cwt. and the truck?

8. A mass of 56 lbs. is placed in a swing the ropes of which are 10 ft. long; find the difference in the tension of the ropes when the swing is at its lowest point, (i) when it is at rest, (ii) when it is moving with 20 velos (neglecting the mass of the swing itself).

9. A horse weighing 5 cwt. is going along a road at the rate of 10 miles an hour and passes over a depression in the road whose section is a circle of 40 feet radius; what is the difference between his weight and the average pressure of his feet on the road when at the lowest point of the depression?

10. Two masses of 3 lbs. and 4 lbs. on a smooth horizontal plane are fastened together by a light inelastic string 7 ft. long and are each describing a circle so that one point in the string is at rest on the plane; what must be the point in the string?

11. Two masses A and B of 1 lb. each are joined by a string; the mass A describes a circle of radius 2 ft. with uniform speed on the surface of a smooth horizontal table, while the other mass B is suspended under the action of gravity by the string, which passes through a small hole in the table at the centre of the circle; find the speed of A.

12. Two masses A and B are joined by a string; the mass A of 10 lbs. describes on a smooth horizontal table a circle of radius 5 ft. with uniform speed 5 velos; the other mass B is suspended under the action of gravity by the string which passes through a small hole in the table at the centre of A's circle; find the mass of B that it may rest in equilibrium.

13. A particle of 3 lbs. is fastened to a fixed point by a string $2\sqrt{2}$ feet long and is describing a horizontal circle under the action of gravity, so that the string describes a right circular cone, the inclination of the string to the vertical being 45^0; prove that the velocity of the particle is about 8 velos.

14. Shew that the tension of the string in Question **13** is $3\sqrt{2}$ lbs. weight.

15. A particle of m lbs. is fastened to a fixed point by a string l feet long, and is describing a horizontal circle under the action of gravity, so that the string describes a right circular cone of vertical angle 2α; prove that the velocity of the particle is

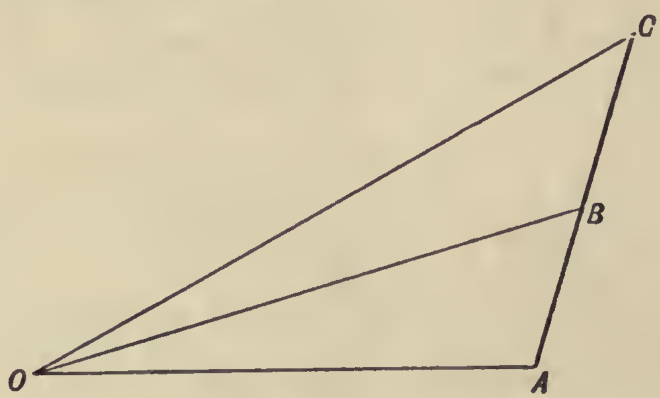
$$\sqrt{(lg \sin^2 \alpha \sec \alpha)} \text{ velos.}$$

NOTE. The arrangement described in Question **15** is called a *Conical Pendulum.*

16. One end of a string $2l$ feet long is fastened to a point A on a fixed smooth vertical rod, the other to a small ring P of mass m lbs. which slides on the rod; another mass Q m' lbs. is fastened to the middle point of the string and revolves with velocity v velos in a horizontal circle so that the angle AQP is a right angle, prove that

$$v^2 = \frac{lg}{\sqrt{2}} \frac{(m' + 2m)}{m'}.$$

145. PROP. *When a point* P *is moving with uniform velocity, the line* OP *joining* P *to any fixed point* O, *traces out equal areas in equal intervals.*



Let the point P be moving in the straight line ABC with uniform velocity; P passes over equal distances in equal intervals; let AB and BC be two distances passed over by P in any two equal intervals; then, $AB = BC$.

Therefore the area $OAB = OBC$.

But the areas OAB, OBC are areas traced out by OP in any two equal intervals; so that OP traces out equal areas in equal intervals. Q. E. D.

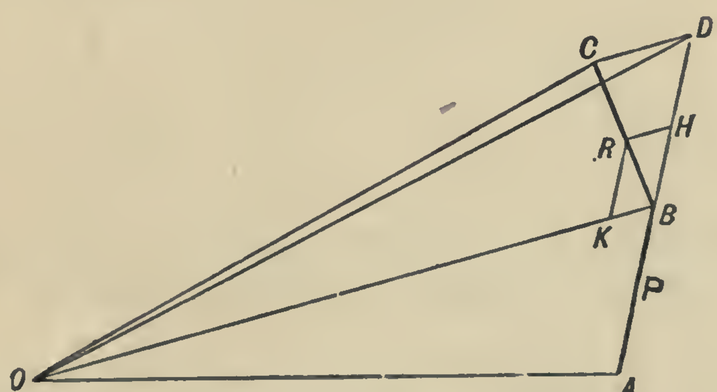
146. PROP. *A point* P *is moving with uniform velocity* u *velos, (so that the line* OP *joining* P *to a fixed point* O *is tracing out equal areas in equal intervals); at a certain instant the velocity of* P *receives a certain addition of* ω *velos, in the direction* PO; *it is required to prove that such an addition of velocity does not alter the rate at which* OP *is tracing out area.*

Let the point P be moving with u velos along AB.

Produce AB to D; let BH represent u velos.

When P is at B let it receive an additional velocity ω velos represented by BK, where BK produced passes through O.

Complete the parallelogram $BKRH$; then the subsequent path of P is along BR, with velocity v velos represented by BR.



Let $AB = ut$ ft.; produce AB to D, making $BD = AB$; through D draw DC parallel to BK; then $BC = vt$ ft.; so that AB and BC are the distances passed over by P in equal intervals of t seconds, just before and just after, the addition of the velocity ω velos. Also AOB and BOC are the areas traced out in the same equal intervals.

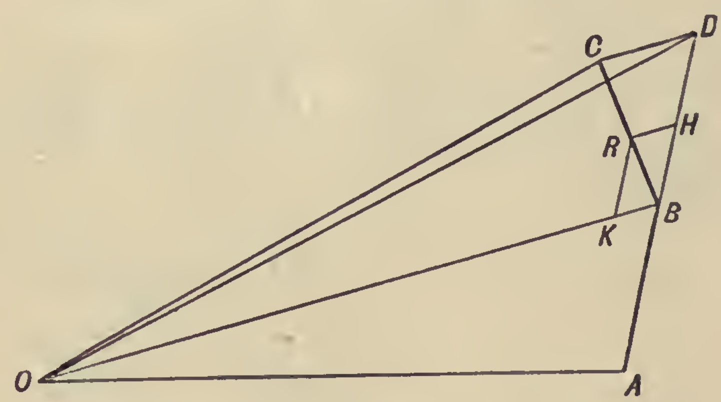
But the area $BOC = BOD$, since CD is parallel to OB; and the area $BOD = AOB$, for $AB = BD$.

Therefore, the area $BOC = AOB$.

That is, the area traced out by OP in any two equal intervals just before and just after the addition of the velocity BK, is the same.

In other words, the rate at which OP traces out area is unchanged. Q. E. D.

147. PROP. The converse of Art. 146. *If a point* P, *moving with uniform velocity* u *velos along* AB, *receive an additional velocity when it is at* B, *in such a manner that the* rate, *at which the line joining* P *to some fixed point* O *traces out* area, *is unchanged, then this additional velocity must be in the direction* BO.



Let AOB and BOC be the areas traced out by OP in equal intervals of t seconds. Produce AB to D, making $BD = AB$.

Let BH represent u velos; let BR represent the resultant velocity; then HR represents the additional velocity; and since

$$BD = AB = ut \text{ and } BC = vt,$$

HR is parallel to DC.

But by hypothesis, the area $OBC = AOB$, and the area $AOB = OBD$. Therefore the area $OBC = OBD$.

Wherefore DC is parallel to BO.

But the additional velocity at B is parallel to DC and therefore is in the direction BO. Q. E. D.

148. It follows from the proposition of Arts. 146 and 147, that when the velocity of a moving point P receives any *series* of additions each of which is such that it is in the direction of the line joining P at that instant to a fixed point O, then the line OP traces out equal areas in equal intervals.

149. And conversely, if the line joining a moving point to a fixed point O traces out areas at a constant rate per second, then whatever addition is made to its velocity that addition must always be in the direction which PO has at that instant.

150. This is true whether the additional velocities are large or small and whether the intervals are large or small.

151. Suppose now that a force of any magnitude acts on a particle P of mass m, so that the force is always in the direction of the line joining P to a fixed point O.

Newton imagines this force to be the limit of a series of small instantaneous impulses acting on the particle at small intervals. These impulses will each be along the line joining P to O at the instant of its action. Accordingly these impulses will generate small additional velocities in the particle, each velocity being *in the direction PO.*

Therefore however small and however frequent these impulses are, the rate at which OP traces out area will be unaltered by them.

Hence, when a particle P is acted on by a *force* which always is directed towards a fixed point O, OP traces out equal areas in equal intervals.

152. The converse is also true. If the motion of a particle P is such, that the line OP joining P to a fixed point O, traces out equal areas in equal intervals, then whatever force acts upon P, the direction of this force must always be towards O.

EXAMPLES. XXXVI.

1. Prove that a mass which is describing a circle with uniform speed, is acted on by a force whose direction tends always towards the centre of the circle.

2. Kepler observed that the line joining the centre of the sun S to any one of the planets P, traces out equal areas in equal intervals; what is the direction of the force under which the planets are moving?

3. Prove that a Planet moves faster when nearest the Sun than when it is furthest from the sun.

CHAPTER XII.

RELATIVE MOTION.

153. When we speak of the *distance* of a point, we must always fix upon another point *from which* that distance is to be measured.

So when we speak of the *velocity* of a point P (that is, of the rate of increase of the distance of the point P), we must again fix upon some point O *from which* the distance which increases is to be measured.

Thus the velocity of a point is a *relative* term only.

154. When the point O, from which the distance of P is measured, is at a distance from another point O', then the distance of P from O' is the resultant of the distances of P from O and of O from O'.

Similarly, when P has velocity relatively to O, and O a velocity relatively to O', then the velocity of P relatively to O' is the resultant of those two velocities.

For example, when we speak of the velocity of a railway train we mean its velocity relative to some point on the earth's surface. The surface of the earth is itself in rapid motion.

155. When a number of points have, besides their own proper motion, a velocity *common* to each of them, this common velocity has no effect on the *relative* distances, after any interval, of the points *from each other*.

156. The same is true of an *acceleration common* to a number of points.

157. Therefore it is also true, when a *force* is applied to each of a system of particles, such that the magnitude of each force is proportional to the mass of the particle to which it is applied (so that each particle has produced in it a certain acceleration).

For example, the *weight* of any particle produces in it a certain fixed acceleration.

Example. A small shell explodes in mid-air, in such a way that each fragment has the same additional initial velocity u velos *away from the centre of the shell, given to it at the instant of explosion; shew that (neglecting the resistance of the air) after any interval* t secs. *the fragments all lie in the circumference of a* sphere of radius ut ft.

Each fragment is moving (i) with the velocity of the shell, say v velos at the instant of the explosion, (ii) with the additional velocity u velos, and (iii) with the acceleration g celos due to gravity, downwards.

To find its position after t seconds we must draw from the initial position O (which is that of the shell at the instant of explosion), a line vt ft. parallel to v, a line $\frac{1}{2}gt^2$ vertically downwards, arriving at a point O'; then from O' a line ut in a different fixed direction for each fragment.

The point O' is the same for each fragment; so that after t seconds *each* fragment is at a distance ut ft. from the same point O'. This point O' is the position which the small shell would have occupied had it not exploded.

EXAMPLES. XXXVII.

1. A shell when moving horizontally explodes so that its mass is equally divided and one part is brought momentarily to rest by the explosion; shew that the velocity of the other part is doubled by the explosion; and if the shell was at the height of 400 ft. and moving with 100 velos, shew that the distance between the two portions on reaching the ground is 1000 feet.

2. A number of guns are fired simultaneously from the same point in different directions each making the same angle with the horizon, the muzzle velocity of the bullets being the same; prove that the bullets are after any interval to be found on the circumference of a horizontal circle.

3. A number of rifles are simultaneously discharged in all directions from the same point, the bullets having each a muzzle velocity of 1000 velos; prove that when the bullet which was projected vertically has reached its highest point, all the bullets are on the surface of a sphere of radius 31250 ft.

4. A number of particles are simultaneously swept off a table with different horizontal velocities; shew that they all reach the ground simultaneously.

5. A series of particles are simultaneously projected along a horizontal line on an inclined plane with different horizontal velocities; shew that as they slide down the plane they continue in the same horizontal line.

6. Shew that if a series of spheres are projected as in Question **5**, which impinge on one another, they will always be in the same horizontal line.

7. Prove that the force which a railway engine exerts in pulling a train, over and above that necessary to overcome the friction, is distributed among the component parts of the train in the proportion of their masses.

8. Shew that if any number of particles are moving each with the same velocity (the same in direction and magnitude) in space, and like parallel forces proportional to the masses of these particles are applied to each of them respectively, the relative positions of the particles will not be altered thereby.

9. A ball is thrown into the air; a man with a rifle aims at it and fires at the instant when the ball appears to his eye to be at rest; shew that if he aims straight at the ball the bullet will hit it.

10. Shew that if the man of Question **9** pulls the trigger at any other instant (his rifle being at rest) the bullet will *not* hit the ball if the rifle be pointed straight at it.

CHAPTER XIII.

THE HODOGRAPH.

158. We represent the path of a moving point P by a line drawn on the paper.

The place from which the point is observed is represented on the paper by a fixed point O; the distance of P from O is represented by the straight line OP.

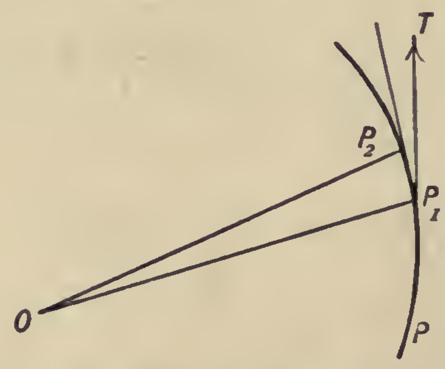
Consider the positions P_1, P_2 which P has at the beginning and at the end of an interval t secs.

The additional distance of P in the interval t secs. is represented by the straight line P_1P_2.

When P is moving with uniform velocity u velos, then $P_1P_2 = ut$ ft.

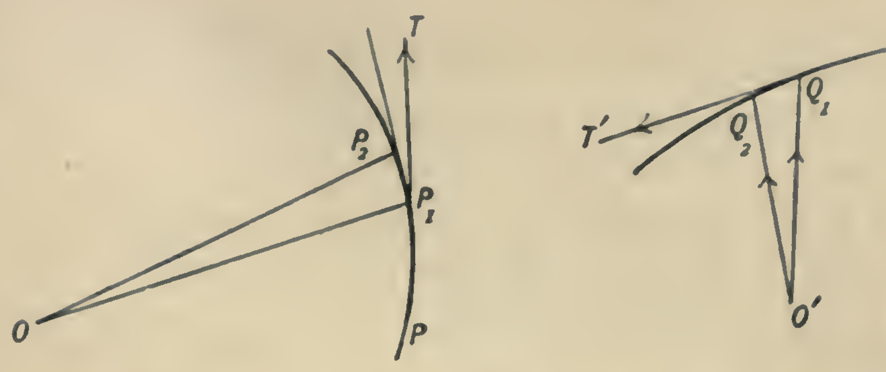
159. When the path of P is a curved line, a tangent to the curve at a point P_1, gives the *direction* of the velocity of P at the instant when the point is at P_1.

Take a point P_2 near to P_1.



Then P_1P_2 is a line whose direction approaches that of the tangent at P_1, when P_2 is made to move nearer to P_1.

Also, if $P_1P_2 = s$ ft., and the point P take t secs. in going from P_1 to P_2, then $\frac{s}{t}$ approaches to the number of velos in the velocity of P when at P_1, as P_2 approaches P_1.



160. Now take another fixed point O'; from O' draw a line $O'Q$ to represent the *velocity* of the moving point P. If we consider Q to move so that the line $O'Q$ is always parallel and proportional to the velocity of P, Q will trace out a line. The line traced in this way by the point Q is called the **Hodograph** of the point P.

PROP. *The* velocity *of the point* Q *on the hodograph represents the* acceleration *of the point* P.

161. On the hodograph take the two points Q_1, Q_2 which correspond to the points P_1, P_2 on the path of P. Then Q_1Q_2 represents the additional velocity added to P in the interval t secs.

For OQ_2 is the velocity at P_2, and it is the resultant of the two velocities OQ_1 and Q_1Q_2. [Art. 99.]

Suppose Q_1 Q_2 represent v velos, then, as P_2 approaches P_1, the ratio $\frac{v}{t}$ approaches the number of celos in the acceleration which P has when at P_1.

Also, the direction of the tangent to the hodograph at Q_1, is the direction of the acceleration of P when at P_1.

In other words, the velocity of Q in the hodograph represents the acceleration of P on the curve.

Example i. The hodograph of a point which moves with uniform velocity is *a point.*

The line OQ is fixed. Q has no velocity.

Example ii. When a point is moving with a velocity which is uniformly increasing in a given direction, the hodograph is a straight line along which it moves with uniform velocity.

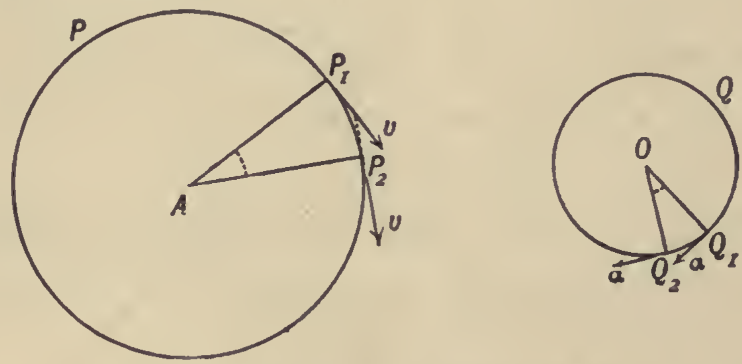
Example iii. When a point is moving with constant *speed* [see Art. 78] in the circumference of a circle, its hodograph is a circle; and the point Q moves along the hodograph with uniform *speed*.

For OQ represents the velocity of P, and it is of constant magnitude and its direction turns through equal angles in equal intervals.

162. PROP. *When a point moves with constant speed* v velos *along the circumference of a circle of radius* r feet, *it has a constant acceleration* $\frac{v^2}{r}$ celos *whose direction always points to the centre of the circle.*

Let the point P be describing the circle PP_1 whose centre is A with constant speed.

Draw the hodograph of P; let this be the circle QQ_1 whose centre is O; then the acceleration of P is represented by the velocity of Q.



Let Q_1, Q_2 be the points on the hodograph corresponding to the points P_1, P_2 on the path of P.

Then OQ_1 is parallel to the velocity of P at P_1 and is therefore perpendicular to AP_1.

The velocity of Q at Q_1 is perpendicular to OQ_1, and is constant. Therefore the acceleration of P at P_1 is in the direction P_1A and is constant.

Now Q describes the circumference of its circle while P describes the circumference of its circle; let the interval in which this is done be t secs.; let the velocity of Q be α velos.

Then, $O'Q = v$ ft. and $\alpha t = 2\pi v$,

also, $vt = 2\pi r$,

whence, $\alpha = \dfrac{v^2}{r}$. Q.E.D.

EXAMPLES. XXXVIII.

1. What is the hodograph of a point which moves with constant speed along the perimeter of a regular polygon?

2. Prove that the hodograph of the path of a projectile is a straight line.

3. Prove that the velocity of a projectile at any point P of its path is proportional in magnitude to SY where Y is the foot of the perpendicular from the focus of its path on the tangent to its path at P.

4. By means of Ex. 3, shew that the hodograph, with reference to any fixed point on the axis, of a projectile is a straight line drawn parallel to the axis at a distance from it equal to the distance of the vertex of the parabola from the focus.

5. Prove that if the tangent to the path of a projectile P be drawn at P to cut the tangent at the vertex at Y, then, as the projectile P traces out its curve, Y moves with uniform velocity.

6. A line is drawn parallel to the axis of a parabola at a distance from it equal to the distance of the vertex from the focus, and a line is drawn from any fixed point on the axis parallel to the tangent at P to cut this line in Q; prove that if P move with the velocity of a projectile describing the parabola, then Q will move with uniform velocity.

SECTION IV.

WORK, ENERGY, POWER.

CHAPTER XIV.

WORK.

163. An external force is said to be doing **work** on a mass when its point of application is passing over *distance* in the direction of the force.

Examples. The weight of a mass falling freely does work on the mass.

The pressure of the foot of a bicyclist does work on the treadle of his bicycle.

The pull of a horse does work on the carriage it is drawing.

When a horse is attached to a railway truck and pulls at it, then as long as the truck is *at rest*, we say the horse is *leaning* against the truck, and does *no work*; but when the truck *moves* and the horse continues his pull, he does work as long as his pull is continued.

164. DEF. The **work** done in an interval on a mass by an external force, varies directly as the force; and also directly as the resolved part in the direction of the force, of the *distance* passed over in that interval by the point of application of the force.

Thus, the work done by a force applied to a mass in motion, is doubled, trebled, etc., when the force is doubled, trebled, etc., for the same distance; and the work done by a given force is doubled, trebled, etc., when the resolved part of the distance in the direction of the force is doubled, trebled, etc.

It is convenient to refer to the resolved part in the direction of the force, of the distance passed over by the point of application of a force, as *the distance through which the force works.*

NOTE. The work done by an external force on a moving mass is *independent of the length of the interval occupied;* it depends only on the *force* and the *distance.*

165. DEF. We shall choose as our **unit** *work* the *work* done by a *poundal* while its point of application moves over a *foot* in the direction of the poundal.

We shall call this *unit* work a **foot-poundal.**

Hence the **measure** of the work done by a uniform force, is the product of the *number* of poundals by the *number* of feet through which it works.

Example. The work done by the *weight* of 1 lb. while the 1 lb. passes over 1 foot vertically downwards, is g foot-poundals.

For, the weight of 1 lb. $= g$ poundals,

1 poundal acting through 1 ft. $=$ 1 foot-poundal,

g poundals acting through 1 ft. $= g$ foot-poundals.

166. When the distance through which the force works, is passed over by the point of application in the direction *opposite* to that of the force, this distance is *negative;* and in this case the work done by the force is said to be **negative**; or we say that work is done *against* the force.

Work which is done in lifting mass under the action of gravity, is done *against* the weight of the mass.

167. The work done against gravity in raising 1 lb. through the vertical distance 1 foot, is g foot-poundals.

This amount of work is called a **foot-pound**, which is an abbreviation for a **foot-pound weight**.

Hence *one foot-pound* is equal to *g foot-poundals.*

Example. Find the work done *by the weight of* 1 cwt. *while the* cwt. *falls a vertical distance of* 10 feet.

The force is 32×112 poundals.

The distance passed over by the mass in the direction of the force, that is, the distance through which the weight works, is 10 ft.

The work done is therefore $\underline{32 \times 112 \times 10 \text{ foot-poundals.}}$

168. When two equal and opposite forces (as, for example, the action and reaction of a stress) act upon the *same* point the work done by one of the forces is equal and opposite to the work done against the other.

Also, if two equal and opposite forces act upon different points, which points have no motion *relative to each other*, the work done by one of the forces is equal and opposite to that done against the other force.

The total work done by two such forces is zero.

Example i. The tension of an inextensible string does no work.

A man in a railway carriage who presses with his feet against the opposite seat does no work, notwithstanding that the mass, to which he applies two equal and opposite forces, is in motion.

Example ii. When the masses in an Atwood's machine are *equal* and are in motion the weight of one does as much work on its mass as the second mass does against its own weight.

Example iii. 1 cwt. *slides down an inclined plane of height* 10 ft. *and inclination* α; *find the* work done *by the* weight *of the mass when the mass goes from the top to the bottom.*

The distance passed over by the mass is the slant side of the plane; the direction of the force is vertically downward; the resolved part of the distance in the direction of the force is the vertical distance passed over; this is, the height of the plane, viz. 10 feet.

Hence, the work done is $10 \times 112 \times 32$ foot-poundals;

that is, 1120 foot-pounds.

Example iv. *A horse draws a tram-car of* 2 tons *along a horizontal road with the uniform velocity of* $7\frac{1}{2}$ miles an hour; *supposing the resistance of the friction etc. is* $\frac{1}{80}$ *of the weight of the car, find how many* foot-pounds *the horse is doing* per second.

The resistance of friction etc. is a force equal to the weight of $\frac{1}{80}$ of 2 tons; that is, the weight of 56 lbs.

Therefore the horse must be applying a force of $56 \times g$ poundals; [for, since the velocity of the car is uniform, no additional force is required by the inertia of the car.]

The point of application of this force is moving at the rate of $7\frac{1}{2}$ miles per hour; that is, of 11 velos.

Therefore the horse is doing $11 \times 56 \times g$ foot-poundals per second;

that is, 616 foot-pounds per second.

169. PROP. *The number of* foot-pounds in the work done *by the* weight of *a mass while the mass is moved from one position to another is the product of the* number of lbs. weight *by the* number of feet *in the vertical distance of the second position below the first.*

For we may consider the path of the mass to be along a number of inclined planes; the work done by the weight in descending any one of these planes is the same as the work done in descending the vertical height [*Example* iii. above].

The work done in *ascending* any inclined plane is that due to its vertical height and is negative; hence

The work done by the weight of a mass in going by any path whatever from one position to another is the same as that done by the weight of the mass in descending from the first position to the horizontal plane in which the second position lies.

Thus, the work done in raising 1 ton through a vertical distance 50 feet is the same, viz. $2240 \times 50 \times g$ foot-poundals, by whatever path the ton arrives at that height. It may be raised by screws; or by a system of pulleys; or by a horse pulling it up an incline in a cart; or it may be raised by a man carrying it piece-meal up a ladder.

EXAMPLES. XXXIX.

Find the number of foot-poundals and of foot-pounds done by the weights of the following.

1. 56 lbs. in descending 40 feet.

2. A man of 10 stone in descending 500 feet.

3. A block of stone of 10 tons in descending 500 feet.

4. A man of 12 stone in ascending a tower 200 ft. high.

5. A train of 300 tons in descending an incline of 1 in 50, one mile in length.

6. A waggon and horses of 3 tons in descending a hill, whose inclination is 1 in 30, for a distance of 200 yards.

7. Compare the work done in 1 hour against gravity (i) by a man who lifts 28 lbs. a height of 3 ft. every minute, (ii) by a man of 10 stone who walks up an incline of 1 in 6 at the rate of 4 miles an hour.

8. Find the work done against friction in moving a ton 100 feet along a rough horizontal plane the horizontal resistance of the friction being $\frac{1}{10}$ of the weight.

9. Find the number of foot-pounds per second required to draw a cart of 1 ton along a horizonal plane uniformly at the rate of 5 miles an hour, the resistance due to friction etc. being $\frac{1}{70}$ of the weight.

10. Find the greatest uniform velocity with which an engine capable of producing 140,000 foot-pounds of work per second, can draw a train of 100 tons along a horizontal plane, the resistance due to friction etc. being $\frac{1}{80}$ of the weight.

11. An engine is drawing a train of 120 tons up a smooth inclined plane of 1 in 60 at the rate of 24 miles an hour; how much work is it doing per second (friction to be neglected)?

12. Supposing the resistance due to friction etc. in Question **11** to be equivalent to the weight of 1120 lbs. and to act opposite to the direction of motion, find at what rate the engine could draw the same train up an incline of 1 in 60, doing the same amount of work per second as before.

13. Supposing the resistance due to friction in Question **12** to be $\frac{1}{240}$th of the *pressure* of the train on the inclined plane, find at what rate the engine could draw the train up the inclined plane with the same expenditure of work per second.

14. Supposing that a man of 12 stone in walking raises his whole weight a vertical distance of 1 inch at every step, and that the length of each step is $2\frac{1}{2}$ feet, find how much work the man does in this way in walking 1 mile.

15. Compare the work done by the man in Question **14** with that done by another man of 12 stone going a mile with uniform speed on a bicycle weighing $\frac{1}{2}$ cwt., the resistance due to friction etc. of the bicycle being $\frac{1}{100}$ of the weight.

CHAPTER XV.

ENERGY.

170. DEF. **Energy** is *capacity for doing work.*

There are two different forms of energy, viz. *Potential Energy* and *Kinetic Energy.*

171. DEF. **Potential** *energy* is the capacity for doing work which a mass has by virtue of its *position.*

A spring when *bent* has energy by reason of its position; for it can do work when changing that position.

Air when compressed can do work on being allowed to expand.

A mass at the top of a tower can do work in its descent to a lower level.

Example. Find the difference between the Potential Energy *of* 1 cwt. *at the* top *of a tower* 100 ft. *high, and that of* 1 cwt. *at the foot of the tower.*

In descending from the top of the tower to the bottom, the *weight* of the 1 cwt. does 100×112 foot-pounds;

Hence, 1 cwt. at the top of a tower 100 ft. high has 100×112 foot-pounds more Potential Energy than 1 cwt. at the foot of the tower.

172. DEF. **Kinetic** *energy* is the capacity for doing work which a mass has by virtue of its mass-velocity.

A bullet when in motion can do work in giving up its velocity.

Water when in motion can do work in turning a water wheel.

The mass of a hammer can do work by virtue of its velocity.

The kinetic energy of a particle having a given velocity is measured by the number of foot-poundals it does in giving up that velocity.

173. PROP. *Prove that the difference between the energy of* m lbs. *having* v velos, *and that of* m lbs. *having* u velos, *is* $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals.

Let a force of p poundals be applied to the m lbs. so that in t seconds its velocity is reduced from v velos to u velos; also suppose that in that interval the mass has a retardation α and passes over s feet in the direction opposite to the force; then, by Art. 26. iii,

$$\alpha s = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2,$$

or,
$$m\alpha s = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2;$$

but $m\alpha$ is the number of poundals required to produce in m lbs. acceleration α; so that $m\alpha = p$.

Therefore
$$ps = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.$$

But ps is the number of foot-poundals done by p poundals which acts on a mass while it passes over s feet in its own direction. That is to say, the mass m lbs. having velocity v can do $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals while its velocity is reduced from v to u. Q.E.D. Hence

174. The kinetic energy of a particle of m lbs. which has v velos is $\frac{1}{2}mv^2$ foot-poundals.

For the mass can do $\frac{1}{2}mv^2$ foot-poundals against a force before it is reduced to rest.

Also, if $\frac{1}{2}mv^2$ foot-poundals of work are done on a particle of mass m lbs. at rest, it will thereupon have v velos.

Example. Find the work done by the engine of a train of 120 tons, *moving the train a* mile *from rest and at the same time giving it a speed of* 30 miles per hour; *the motion being on a horizontal line and the resistance due to friction* $\frac{1}{240}$ of the weight of the train.

The work done against friction is

$$(\tfrac{1}{240} \times 2240 \times 120) \times (1760 \times 3) \text{ foot-pounds}$$
$$= 1120 \times 1760 \times 3 \text{ foot-pounds.}$$

The work done on the inertia, in producing kinetic energy, is

$$\tfrac{1}{2} \times 120 \times 2240 \times (44)^2 \text{ foot-poundals} \qquad \text{[Art. 173.]}$$
$$= \tfrac{1}{32} \times \tfrac{1}{2} \times 120 \times 2240 \times (44)^2 \text{ foot-pounds.}$$

Hence, the whole work done by the engine is

$$(1120 \times 30 \times 176 + 1120 \times 30 \times 2 \times 121) \text{ foot-pounds}$$
$$= 1120 \times 30 \times 418 \text{ foot-pounds.}$$

EXAMPLES. XL.

Find the kinetic energy of each of the six following;

1. A stone of 4 lbs. which has 200 velos.

2. A cannon shot of 8 cwt. which has 2000 velos.

3. A cannon shot of 12 cwt. which has 1600 velos.

4. A train of 400 tons going at the rate of 60 miles an hour.

5. A man of 12 stone running at the rate of 12 miles an hour.

6. A man of 12 stone after falling a distance of 10 feet.

7. Find the work done by the engine of a train of 360 tons, in going 2 miles from rest on horizontal rails, the friction being $\frac{1}{240}$ of the weight, and the speed attained 45 miles per hour.

8. Find the work necessary to raise a train of 100 tons to the top of a mountain pass 500 feet high and at the same time to give it a velocity of 60 miles an hour (neglecting friction etc.).

9. A man has to raise 1 cwt. of bricks 8 feet; he throws them up so that they arrive at a point 8 ft. high with 5 velos; compare the superfluous work done with the necessary work.

10. A tram car of 1 ton is stopped by a brake 10 times in going a mile; the brake stops the car in 11 yards; after each stoppage the car attains a velocity of $7\frac{1}{2}$ miles an hour. Supposing the friction of the rails to be a uniform force of 28 lbs. weight, compare the work done in this journey with the work done in going a mile with uniform velocity. [At the beginning of the mile the car is moving at the rate of $7\frac{1}{2}$ miles an hour.]

11. The expense of moving a train is proportional to the work done. Compare the cost of getting the speed of a train up from rest to a velocity of 45 miles an hour and at the same time going 1 mile, with the cost of moving it a mile with uniform velocity; supposing that the resistance caused by friction etc. to be $\frac{1}{120}$ of the weight of the train.

12. Suppose the tram-car in Question **10** had no brake, and so had to be stopped by the horses; find how much more work the horses would have to do in consequence in a journey which includes 10 stoppages.

175. When a particle slides along a *smooth* **plane** the pressure of the plane, whatever it is, is always perpendicular to the plane, and therefore perpendicular to the path of the particle; the *distance* through which the pressure *works* is therefore zero; hence

176. The pressure of a smooth plane on a particle moving along its surface does no work on the particle.

Example. When a particle slides down a smooth inclined plane under the action of gravity only, its total energy *is unchanged.*

By *Example* i. p. 111, a particle of m lbs. sliding down an inclined plane of angle α has $g \sin \alpha$ celos.

Let h feet be the *vertical* distance passed over by the particle in any interval t seconds during which the velocity of the particle is changed from u velos to v velos.

Then the particle has lost mgh foot-poundals of potential energy and gained $(\frac{1}{2}mv^2 - \frac{1}{2}mu^2)$ foot-poundals of kinetic energy; we have to prove that these amounts of work are equal.

Let s feet be the distance passed over on the plane,

then $$s(g \sin \alpha) = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2,$$

but $$h = s \sin \alpha.$$

Therefore $$hg = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2.$$ Q. E. D.

177. The sum of the potential energy and of the kinetic energy of a particle is unaltered by any motion under the action of gravity only.

Example i. Let m lbs. fall from rest h feet *vertically*. Then the potential energy is diminished by mh foot-pounds.

It receives v velos, where $mgh = \frac{1}{2}mv^2$; so that it has received kinetic energy capable of doing mgh foot-poundals, that is, mh foot-pounds; in other words, the potential energy lost is equal to the kinetic energy gained.

Example ii. Since the velocity of a projectile is equal to that due to a fall from the directrix, it follows that (the kinetic energy + the potential energy) of a projectile is always equal to the potential energy of an equal particle at rest at some point on the directrix.

178. When a particle slides along a **curved surface** which is *smooth*, it is understood that the pressure of the surface on the particle is always perpendicular to the surface, and therefore this pressure is *always* perpendicular to the path of the particle.

The *distance* through which the pressure works is therefore *always* zero; so that the *work* done by the pressure is zero; hence,

179. *When a particle is sliding on a smooth surface under the action of no forces, its kinetic energy is constant;* consequently *its speed is constant.*

*180. We may also prove this important proposition as follows:

PROP. *An inelastic particle slides along the smooth sides of a regular polygon; prove that, in the limit when the polygon becomes a circle, the impacts at the angular points have no effect on its speed.*

Let AB, BC be two of the sides of the regular polygon. Produce AB to K.

Let the velocity along AB be v velos.

Resolve this velocity along and perpendicular to BC; the resolved parts are $v \cos \alpha$ velos and $v \sin \alpha$ velos.

The resolved part $v \sin \alpha$ velos is destroyed by the impact.

Hence after impact the particle will have $v \cos \alpha$ velos along BC.

Similarly after the next impact it will have $(v \cos \alpha) \cos \alpha$ velos along the next side and so on.

Hence, when the velocity of the particle is turned through a finite angle A, where $n\alpha = A$, it will have $v \cos^n \alpha$ velos,

that is $$v \left(\cos \frac{A}{n}\right)^n \text{velos}.$$

Now let n be increased and α diminished without limit so that A is kept unchanged; then the limit of $\left(\cos\frac{A}{n}\right)^n$ is 1.†

Hence, when the polygon becomes a circle, $v\,(\cos\alpha)^n = v$.

So that the speed of the particle is unchanged, when its direction has turned through a finite angle A. Q. E. D.

*181. Since the above proposition is true of a circle of *any* radius, and of an arc of any magnitude, it must be true for *any* continuous curve.

For such a curve may be considered to be made up of successive small arcs of its successive circles of curvature.

182. When a particle slides on a smooth continuously curved surface, under the action of gravity only, its total energy is constant.

For, the only forces acting are the pressure of the surface and the weight of the particle; and the pressure of the surface does not alter the kinetic energy of the particle, while the weight of the particle produces its own effect, no matter what other forces may be acting. The effect of the weight is expressed by saying that the total energy of the particle is constant.

Example i. *A smooth fine wire, in the form of a circle of radius* r ft. *is placed in a vertical position, and a particle in the form of a small ring slides on it under the action of gravity. The ring has* u velos *when at the highest point of the circle; find its* velocity at the lowest point.

Let the ring contain m lbs.

In going from the highest point to the lowest it loses

$2\,rmg$ foot-poundals of potential energy.

Let its velocity at the lowest point be v velos; it will therefore have gained $(\frac{1}{2}\,mv^2 - \frac{1}{2}\,mu^2)$ foot-poundals of kinetic energy.

But on the whole, its energy is unchanged.

Therefore, $$2\,rmg = (\tfrac{1}{2}\,mv^2 - \tfrac{1}{2}\,mu^2),$$

or, $$\tfrac{1}{2}\,mv^2 = \tfrac{1}{2}\,mu^2 + 2\,rmg.$$

† Lock's Higher Trigonometry, Art. 9.

Example ii. *A stone is fastened by a light inextensible string* 1 foot *long, to a fixed point, and is describing a vertical circle about the fixed point as centre, under the action of gravity; find the* least possible velocity at the lowest point of the circle.

The constraint of the string is similar to that of a smooth circular wire; for the tension of the string, being always perpendicular to the direction of motion, does no work; hence, by Art. 179, the total energy of the stone is constant.

Therefore, if v velos be the velocity of the stone at the lowest point of the circle, u velos its velocity at a point on the circle whose vertical distance above the lowest point is h feet, then

$$mgh = (\tfrac{1}{2}\,mv^2 - \tfrac{1}{2}mu^2),$$

or, $$\tfrac{1}{2}v^2 - \tfrac{1}{2}u^2 = gh \quad \text{.................................(i).}$$

Now, in order that the stone may completely describe the circle, the tension of the string (which is *least* at the highest point) must just vanish at that highest point.

Let u velos be the velocity of the particle at the highest point, then its acceleration is

$$\frac{u^2}{r} \text{ celos vertically downwards.}$$

The only force acting when the stone is at the highest point of the circle, is the weight, mg poundals vertically downwards; for the only other force available is the tension of the string, which at this moment is zero.

Hence, $$mg = \frac{mu^2}{r}$$

$$= mu^2 \,[\text{for } r = 1] \quad \text{.....................(ii).}$$

Therefore, since v is the velocity at the lowest point of the circle, putting $h = 2$ ft. in (i) and combining with (ii), we have

$$v^2 = u^2 + 4g = g + 4g = 5g.$$

Hence, the least velocity at the highest point is about $4\sqrt{2}$ velos, and the corresponding velocity at the lowest point about $4\sqrt{10}$ velos.

NOTE. The unit of work in the **C.G.S.** system of units [see page 77] is the work done by a Dyne working through a Centimetre.

This unit work is called an **Erg**.

EXAMPLES. XLI.

1. A particle is describing a parabola under the action of gravity, prove that its total energy is constant.

2. A heavy particle slides down a smooth straight tube from rest and then describes a parabola; find its velocity when it is 100 feet below the point from which it started.

3. A perfectly elastic particle is let fall on a horizontal pavement, and rebounds; prove that the total energy of the particle is constant.

4. Two perfectly elastic spheres impinge directly, prove that their total energy before and after impact is the same.

5. Two perfectly elastic spheres impinge obliquely, prove that their total energy is unaltered by the impact.

6. An imperfectly elastic particle falls on a horizontal plane and rebounds; shew that it loses energy by the impact.

7. Two imperfectly elastic balls m and m', of elasticity e and velocities u velos, u' velos impinge directly; find the energy lost by impact.

8. A multitude of small perfectly elastic spheres are in motion in a closed space surrounded by fixed walls, under the action of no force; prove that the total kinetic energy of the spheres is constant.

9. If the spheres in Question 8 are under the action of gravity, shew that their total energy is constant.

NOTE. A **simple pendulum** consists of a heavy particle fastened to a fixed point by a light rod; the particle moves in the lowest arc of a vertical circle under the action of gravity.

10. A simple pendulum *of length* l feet swings through an arc 4α, prove that its velocity at the lowest point of its path is $2 \sin\alpha \sqrt{(gl)}$ velos.

11. A circular wheel, whose rim consists of a heavy uniform wire of m lbs. and whose spokes, (of length r feet), and axle are such that their mass may be neglected, revolves about its axis, which is fixed, so as to make n complete revolutions per second; shew that the kinetic energy of the wheel is $2mn^2\pi^2r^2$ foot poundals.

12. A fine light string is coiled round the wheel of Question 11, one end being fastened to the rim and the other end supporting a heavy mass m' lbs. This mass descends from rest under the action of gravity, causing the wheel to turn as it descends; prove that when m' has descended h feet from rest its velocity is

$$\sqrt{\frac{2m'gh}{m+m'}} \text{ velos.}$$

CHAPTER XVI.

POWER.

183. We use the word '**agent**' to denote a *machine* which transforms some stored-up chemical or other energy into work.

Examples. A *navvy* transforms some of the chemical energy of his food into work when he raises an embankment.

A *steam-engine* transforms some of the energy stored in coal into work when it does work on a train.

A *windmill* transforms some of the kinetic energy of the air into work when it pumps up water, etc.

184. DEF. The **power** of an agent varies as the amount of work which the agent can produce from the energy supplied to it in a unit of time.

The **unit** *power* is the power of an agent which can transform 1 foot-poundal per second.

Hence the **measure** of the power of an agent is the *number* of foot-poundals which it can transform per second.

185. A **Horse-power** is the *power* of an agent which can transform 550 foot-pounds per second;

that is, $550 \times g$ foot-poundals per second.

Example i. *At what uniform speed can a* horse *of* 1 horse-power *draw a tram-car of* 1 ton, *supposing the friction etc. to cause a uniform resistance of* 50 lbs. weight?

Suppose the required speed to be v velos.

Then the tram passes over v feet per second.

Therefore the horse must produce $v \times 50$ foot pounds per second.

He can produce 550 foot pounds per second.

Hence $v = 11$.

That is, he can draw the car at the rate of 11 velos;

or, at the rate of $7\frac{1}{2}$ miles per hour.

Example ii. *At what uniform* speed *can a man of* $\frac{1}{10}$ horse-power, *draw* 1 ton *up a smooth inclined plane of* 1 in 10?

The sine of the angle of inclination is $\frac{1}{10}$.

Therefore the force necessary to move the ton up the plane is $\frac{1}{10}$ of a ton weight; or, 224 lbs. weight.

[Now it is very unlikely that the man can exert such a force, unaided.

To do so, he must make use of a wheel and axle, or some such mechanical contrivance, which by multiplying the distance through which the muscular force of the man works, divides by the same number the force which must be exerted.]

Let the velocity required be v velos,

then, $$v \times 224 = 55,$$

or, $$v = \frac{55}{224} = \frac{1}{4} \text{ (very nearly).}$$

That is, he can (by the aid of a machine) draw the ton up the plane at the rate of about 15 feet per minute.

186. The following is an illustration of the meaning of the terms work, energy, power.

Suppose a Contractor undertakes to excavate a certain amount of earth to form a canal or dock. The raising of the earth and the moving it from one place to another requires the doing of a certain amount of work against gravity and against friction. He will bring together a certain number of labourers and steam-engines, and these together are the *power* he proposes to use. But no matter how much power he brings together he can do none of the work (that is, he can excavate no earth) until he has brought up energy in the form of food and fuel. He may use as much power as he pleases and so get the work done quickly or slowly according as the amount of the power he provides is greater or less. But he must provide energy (stored either in food or in fuel) in proportion to the amount of the work he has undertaken to do.

It must not be supposed however that a steam-engine produces in the form of useful work more than a small proportion of the energy stored in the coal or other fuel with which it is fed.

A man is a much more **efficient** engine than a steam-engine; that is, a man can produce in the form of useful work a much larger proportion of the energy stored in his food than the steam-engine can of the energy stored in its fuel. But a steam-engine is a cheaper worker than a man because energy in the form of coal is much cheaper than energy in the form of food.

187. DEF. The number of foot-pounds of work yielded by a steam-engine in consequence of the consumption of 1 bushel (*i.e.* 84 lbs.) of coal is called the **duty** of the engine.

The average duty of a good low-pressure steam-engine is about 90 millions.

EXAMPLES. XLII.

1. What must be the power of an engine that can keep a train of 100 tons going at the rate of 60 miles an hour on horizontal rails when the friction is equivalent to a horizontal resistance of $\frac{1}{80}$ of the weight of the train?

2. At what uniform rate could the engine of Question 1 draw the same train up an incline of 1 in 50, the resistance of friction etc. being unaltered?

3. In how many minutes would an engine of 5 horse-power raise 12 cwt. a vertical height 192 feet?

4. What uniform velocity can an engine of 60 horse-power maintain in a train of 90 tons against resistances of friction etc. of 8 lbs. (weight) per ton?

5. A horse can draw a cart of 1 ton at the rate of 4 miles an hour; supposing that his power is 1 horse-power, find the resistance of friction.

6. What is the weight of a train in which the engine of 50 horse-power can maintain a uniform velocity of 25 miles per hour against resistances of 8 lbs. wt. per ton?

7. What must be the power of an engine which can raise 1000 gallons of water per minute a height of 20 feet, delivering it by means of a uniform pipe of which a linear foot contains 1 gallon? [*A pint of water weighs a pound and a quarter.*]

8. A man throws a brick of 7 lbs. a height of 8 feet once in a second; at what rate is he working?

9. The length of the stroke of the piston of an engine is 7 ft. 4 in.; the mean pressure of the steam is 12 lbs. (wt.) per sq. inch; the number of strokes made by the piston is 20 per minute; what must be the surface of the piston in order that the horse-power of the engine may be 80?

10. The internal diameter of the cylinder of a steam-engine is 1 ft. 6 in.; the length of the stroke is 4 ft.; it makes 18 strokes per minute; under what average pressure of steam per square inch must it work in order that work at the rate of 75 horse-power may be done on the piston?

11. It is found that a certain water-wheel gives out in useful work ·55 of the potential energy lost by the water in passing over it. If the average section of the stream is 5 ft. by 2 ft.; its velocity 1 velo; and a fall at the wheel of 12 ft.; what is the horse-power of the machine?

12. How many bushels of coal must be expended per hour in raising 600 cubic feet of water per minute a height of 850 feet, the duty of the engine being 75 millions?

CHAPTER XVII.

ENERGY IS INDESTRUCTIBLE.

188. When a machine does work it receives a supply of some kind of energy which it transforms into mechanical work.

A man who raises weights receives a supply of energy by the chemical combustion of the food which he eats.

189. A machine which has no energy supplied to it cannot do work; it can only *transmit* work.

A system of pulleys used in raising weights transmits the work given to it by what is termed, by many writers on Statics, '*the power*'; this work is transmitted to what is called '*the weight*.'

It will be found that (when the kinetic energy is unchanged and friction is neglected) the work done by '*the power*' is always equal to the work done on '*the weight*.'

For example, if the weight is 16 times the power, the power will have to do work for a distance of 16 feet in raising the weight 1 foot.

190. Since in a *frictionless* machine, whose parts are moving with constant speed, the amount of work put into the machine is equal to the amount of work which the machine gives out, we can often find the ratio of the *power* to the *weight* by the following method.

Suppose it can be ascertained through what distance (d) the '*power*' works in a given short interval of time, and also through what distance (d') work is done on the weight in the same interval; then if the power be p poundals and the weight Q poundals, the work done by the power is $P.d$ *foot-poundals;* the work done on the weight is $Q.d'$ *foot-poundals;* therefore $p.d=Q.d'$; whence, $\frac{p}{Q}=\frac{d'}{d}$.

191. When the machine is such that the ratio of the Power to the Weight is constant, this ratio will be independent of the interval observed. When the ratio of the Power and the Weight varies (*the kinetic energy of the*

machine being constant) then the result obtained above will be more and more nearly true at a given instant, the smaller the interval during which the distances d, d' are observed.

Hence the ratio of the power to the weight is the limiting value of the ratio of the distances d, d' when the interval of observation is infinitely diminished.

192. When the kinetic energy is 'virtually' zero the method becomes a Statical Principle of great importance, and is called **the Principle of Virtual Work.**

193. No machine however can be made in which the resistance of *friction* is entirely excluded; so that no machine can transmit the *whole* of the work put into it.

This friction, which absorbs work, uses the work it absorbs in heating the parts of the machine; it changes some of the work put into the machine into another form of energy which we call *heat.*

194. Energy is indestructible; therefore energy cannot be produced out of nothing; for that which can be produced can be destroyed. One form of energy can be obtained from another; but it can be obtained in no other way.

A steam engine does work; it is supplied with the energy of heat obtained from the combustion of coal; in which coal is stored the energy radiated from the sun countless years ago.

195. Mechanical work involves the moving of the surface of one mass over that of another. This motion invariably sets up friction which absorbs energy, converting it into heat. Heat tends to diffuse itself in all directions. And since heat can only be converted into mechanical energy when the heat is highly *concentrated*, it follows that the energy absorbed by friction is dissipated in such a way that it is lost for all practical purposes.

196. Hence, notwithstanding that energy is indestructible, yet kinetic energy is continually being eaten away, as

it were, by friction and converted into the unserviceable form of uniformly diffused heat.

For example, it is proved in Art. 133 that the kinetic energy of two masses is diminished by their mutual impact. The reason of this loss of kinetic energy is that during an impact the particles of the masses in the neighbourhood of the point of impact are first compressed and then recover more or less their original position. This compression and recovery generates *heat*; accordingly the kinetic energy of the two masses is diminished by the amount of energy taken up by the heat generated.

197. Heat is itself the kinetic energy of the minute particles of a substance.

Gas consists of minute invisible particles of matter; these particles are in motion; they impinge repeatedly and ceaselessly on one another. They are perfectly elastic (possibly because they are too small to be capable of compression); consequently by Art. 133 the joint energy of two particles is unaltered by their impact.

The quantity of heat contained in a certain amount of gas is *the kinetic energy* of all its particles. This energy (if none be allowed to pass away from the gas) continues unaltered because the particles of the gas are perfectly elastic.

198. The main principle which it is the object of Dynamics to explain and establish may be summed up in the words **Energy is indestructible**.

'The discussion of the various forms of energy—gravi-'tational, electro-magnetic, molecular, thermal &c.—with 'the conditions of the transference of energy from one form 'to another, and the constant dissipation of the energy 'available for producing work, constitutes the whole of 'physical science,—in so far as it has been developed in 'the dynamical form,—under the various designations of 'Astronomy, Electricity, Magnetism, Optics, Theory of the 'Physical states of bodies, Thermo-dynamics and Chemistry.' *Professor Maxwell's Matter and Motion*, p. 95.

MISCELLANEOUS PROBLEMS. XLIII.

1. A point moving with a constant acceleration moves over a distance of 100 ft. in a certain minute and moves over a distance of 200 ft. in the fifth minute after that; find its acceleration.

2. A point moving with uniformly increasing velocity is observed to traverse 200 yards in a certain interval of 10 seconds; in the next interval of 8 seconds it is observed to traverse 200 yards; find its velocity at the beginning of the interval of 10 seconds.

3. In a certain interval of 10 seconds a point passes over 220 feet; in the next interval of 5 seconds it passes over 330 feet; assuming that the point is moving with uniform acceleration find its velocity at the beginning of each of the two intervals.

4. A hotel lift moving without friction is observed to descend 4 feet from rest in two seconds; supposing that the lift with its contents is 2 tons and the tension of the chain supporting it is uniform find that tension.

Supposing the lift to descend with the tension for 30 seconds, and that at the end of that time the tension is increased to the weight of 2 tons 1 cwt., when will the lift come to rest?

5. A mass of 488 grammes is fastened to one end of a cord which passes over a smooth pulley. What mass must be attached to the other end in order that the 488 grammes may rise through a height of 200 centimetres in 10 secs.? ($g=980$ centimetres per sec. per sec.)

6. What is the weight of 1 cwt. at a place where a falling body moves 12 feet from rest in one second?

7. In an Atwood's machine with a 100-gramme weight on one side and an 80-gramme weight on the other, the heavier weight descends 2·18 metres from rest in 2 seconds, prove that $g=981$ centimetres per sec. per sec.

8. A colliery engine winds a cage of 2 tons up a coal shaft at the uniform rate of 10 feet per second; if the tension of the rope is equal to the weight of 2 tons 1 cwt. what is the resistance of friction? Suppose that by the aid of a lubricant the friction on the above is diminished by one half, what will be the motion of the cage, other circumstances being unchanged?

9. Two particles each of mass m are at rest side by side; at a certain instant one receives an impulse B in a given direction, and at the same instant a constant force F is applied continuously in the same direction. If after traversing a distance s in the interval t, they are again side by side, then $2B = Ft$ and $2B^2 = mFs$.

10. With what velocity must a particle be projected vertically that it may rise to the height of 200 feet in 3 seconds?

11. The engine of a train of 100 tons exerts on a horizontal railway a horizontal force equal to the weight of 19 cwt., the retarding force of friction being equal to the weight of 9 cwt. Starting from rest, how long will the train take to attain a velocity of 60 miles an hour?

12. A train of 80 tons on a horizontal railway whose engine exerts a horizontal force equal to the weight of 27 cwt., is observed to move from rest over a distance of half a mile in two minutes, what is the retarding force of friction?

13. A train on a horizontal railway is observed at a certain instant to have a velocity of 10 miles an hour; 3 minutes later it is observed to have a velocity of 25 miles an hour; if the engine was exerting during that interval a horizontal force equal to the weight of 11 cwt., over and above that necessary to overcome the friction, what is the mass of the train?

14. A tram-car of 1 ton on horizontal rails attains a velocity of $7\frac{1}{2}$ miles an hour after going 11 yds. from rest, the retarding force of friction being uniform and equal to the weight of 56 lbs.; find the force, supposed uniform, exerted by the horse on the car.

15. In Question **14**, if the horse weighs 5 cwt., find the horizontal force obtained by the horse from the friction of his feet on the road.

16. It is observed that a horse, whose weight is 5 cwt., in pulling a truck of 5 tons on horizontal rails causes it to move 20 ft. from rest in 30 seconds; the same horse is observed to cause another truck to move 12 ft. from rest in 45 seconds; supposing the horse obtains the same force from the friction of his feet with the ground in each case, find the mass of the second truck (neglecting the friction of the rails, etc.).

17. A mass of 2 lbs. is observed to fall from rest under the action of gravity a vertical distance of 1 ft. in 1 second; what vertical force is acting upon it?

18. A mass under the action of gravity is acted on by a vertical force upwards of 20 poundals, and it is observed to fall a vertical distance of 3 ft. in 2 seconds from rest; what is the mass?

19. A man of 12 stone in a 'lift' stands on a spring which indicates that he is applying to it a vertical pressure equivalent to the weight of 10 stone; what is the motion of the lift?

20. If in Question **19** the spring indicates a pressure of 13 stone, what is then the motion of the lift?

21. A stone is let fall from a height of 16 ft., and alights on a coiled spring which brings it to rest after passing over 1 ft.; the spring is so arranged that the force which it applies to the stone is constant; and in this case the force is equal to the weight of 20 lbs.; what is the mass of the stone?

22. Determine the shortest length in which a barge of 50 tons moving at 3 miles an hour can be brought up by a chain passed round a post, supposing the maximum weight which the chain can support is 1 ton, neglecting any friction which may be retarding the barge.

23. "A man striking with a hammer of 18 lbs. started a bolt "$\frac{1}{8}$th of an inch at each stroke; it required a pressure of 107 "tons to press the bolt down by quiet pressure." With what velocity did the head of the hammer strike the bolt?

24. A ship sailing North-East through a current running 4 miles an hour, finds after 2 hours she has made good 4 miles South-East; determine the direction of the current and the speed of the ship.

25. Determine the position to take in a horizontal road in order to throw a ball over a tower of given height (h) above the road and depth (k) with the least exertion; determine the focus and directrix of the trajectory.

26. Determine the charge of powder required to send a 68-lb. shot with an elevation of 15° to a range of 3000 yards, having given that the velocity communicated to the same shot by a charge of 10 lbs. of powder is 1,600 feet per second.

27. A railway train of n tons moving at the rate of v miles per hour comes to an incline of 1 in m; what additional pull in tons weight will be required from the engine to keep the train moving at the same rate; and what additional horse-power must it exert?

28. The inclination of a smooth inclined plane is 1 in 20 and a particle took 10 seconds to slide down it. How far did the particle slide?

29. Determine in tons the greatest load which an engine whose pull is $20 \times 7 \times 21$ pounds weight can take on a level railway from rest at one station to stop at the next station 33 miles off in 22 minutes, supposing the average resistance of the road on the level to be 14 lbs. weight per ton, the break-power to be 14 lbs. weight per ton, and the engine and tender to weigh 20 tons.

30. Determine in tons the greatest load which an engine whose pull is p pounds weight can take along a horizontal railway from rest at one station to stop at the next station s miles off in t minutes, supposing that the average resistance of the road to be r pounds weight per ton, and that the break-power is b pounds weight per ton. Shew that the greatest velocity attained is $\frac{120s}{t}$ miles per hour; that the driver must shut off steam and apply the breaks $\left\{t - \frac{176 \times 224 \times s}{(r+b)gt}\right\}$ minutes from the time of starting; and that the load is $\frac{pg\tau}{176 \times 2240s + 60bgt\tau}$ tons where τ is the above number of minutes.

31. On an inclined plane in a slate quarry whose length is $13l$ feet and height l feet, each full waggon weighing m tons is made by means of a rope and of a fixed pulley to draw up an empty one of n tons; find the time of descent and the tension of the rope.

32. P and Q are equal masses each n oz. connected by a string passing over a fixed pulley; what weight must be added to one of them that it may descend 1 foot from rest in two seconds, supposing no inertia in the string or the pulley?

33. If any right-angled triangle be placed with its hypotenuse vertical the times of descent from rest down each of the other sides are equal.

34. A body slides down a smooth inclined plane of given height; prove that the time of descent varies as the length of the plane.

35. Given the base of a plane find its height so that the horizontal component of the velocity acquired by sliding down it may be the greatest possible. [Result: height = base.]

36. A ball is dropped from the masthead of a ship sailing at the rate of n miles per hour. Shew that when it has fallen $\frac{121}{3600}n^2$ feet its velocity relative to the earth is inclined to the horizon at an angle 45°.

37. Prove that at the equator a shot fired westward with velocity 8333 or eastward with velocity 7407 metres per second, will, if unresisted, move horizontally round the earth in one hour and twenty minutes and one hour and a half respectively, given that a quadrant of the earth's equatorial circumference is 10^9 centimetres.

38. Prove that if the angular velocity of the earth about its own axis were increased so as to be 17 times as great as it is, then mass would have no weight at the equator.

39. An ounce being taken as the unit of mass, a second as the unit of time and an inch as the unit of length, compare the unit of force with the weight of 1 lb.

40. A simple pendulum is pulled aside until its bob is raised 4 inches and then let go, find the greatest velocity it will have on the subsequent oscillations.

41. A projectile of 1 oz. is started with a velocity of 800 feet per second and arrives at a point in the same horizontal plane after travelling 800 yards with a velocity 600 feet per second; what was the average resistance of the air?

42. A projectile of $\frac{1}{2}$ oz. is started vertically upwards with the velocity 128 feet per second and arrives at a point 16 feet higher with the velocity 64 feet per second; what is the average resistance of the air?

43. Prove that when a projectile passes through two given points A, B the least velocity which it can have when passing through A is that due to a fall through the vertical height equal to half the sum of (AB and the height of B above A).

44. A body is projected up a smooth inclined plane whose height is one-half its length with a velocity of 60 feet per second and just reaches the top; find the length of the plane and the time taken in the ascent.

45. A body slides down a rough plane whose angle is $\tan^{-1}\frac{3}{4}$ a distance of 40 feet along the plane. If the coefficient of friction is $\frac{1}{5}$, find the velocity gained.

46. The head of a nail is struck a direct vertical blow by a hammer of 4 lbs. which at the instant of striking is moving at the rate of 20 feet per second. Supposing the nail to be driven into wood a distance of half an inch and the stress between the hammer and the nail to be uniform, find its magnitude.

47. A fixed forty-ton gun projects a bolt of 5 cwt. with a muzzle velocity of 2000 feet per second; the base of the shot contains 300 square inches; assuming that the interval occupied by the bolt in traversing the tube of the gun is $\frac{1}{9}$ second, find the pressure per square inch on the powder chamber on the supposition that the pressure is constant and that it is four-thirds of what it would be if there were no friction between the tube and the bolt.

48. Prove that the time of flight of a projectile projected with velocity v at the angle α from a point on an inclined plane of angle β, so as to strike the plane again at right angles, is

$$\frac{v \cos(\alpha - \beta)}{g \sin \beta}.$$

49. A particle of elasticity e is projected from a point O in a horizontal plane with velocity v at an angle α; prove that the distance of the point of the nth impact from O is

$$\frac{v^2 \sin 2\alpha (1 - e^n)}{g(1-e)}$$

and that it takes place after the interval $\dfrac{2v \sin \alpha (1 - e^n)}{g(1-e)}$.

50. Two heavy particles are projected from the same point in different directions in the same vertical plane with velocities u, v and elevations α, β respectively; shew that the line joining them will pass through the point of projection after the interval

$$\frac{2}{g} \times \frac{uv \sin(\beta - \alpha)}{u \cos \alpha - v \cos \beta}.$$

51. Prove that when two heavy particles are projected in the same vertical plane from two fixed points with equal velocities so that they collide, the sum of their angles of projection must be constant.

52. It is said that a train moving on straight horizontal rails at the rate of 60 miles an hour can be stopped by means of continuous brakes in less than a quarter of a mile. What must be the average coefficient of friction supposed uniform?

53. A bird of 2 lbs. flying horizontally due southwards at the rate of 33 feet per second at a height of 128 feet from the ground is struck by a bullet of 1 oz. moving vertically at the rate of 1056 feet per second; the bird is killed and the bullet remains imbedded in its body; neglecting the resistance of the air, prove that directly after being struck the bird is moving at the angle 45° with the vertical and that it will reach the ground at a point distant 128 feet horizontally from the point at which it was struck.

54. Prove that the maximum range for a given velocity upon an inclined plane is obtained when the projectile is projected in the direction bisecting the angle between the vertical and the plane; and that the ranges for directions equally inclined to this direction are equal.

55. If a ball be projected with a given velocity from a point in an inclined plane in the direction such that the range on the plane is the greatest, shew that it strikes the plane in the direction perpendicular to the direction of projection.

56. Two equal inclined planes of angle a and height h are placed back to back and a particle is projected with velocity $\sqrt{(3gh)}$ directly up one of them; shew that it will strike the other plane exactly at its foot.

57. A heavy perfectly elastic particle is dropped from a point within a smooth fixed sphere which is distant horizontally $\frac{1}{2}\sqrt{(3-\sqrt{2})}$ times the radius from the centre; prove that it will retrace its path after two rebounds.

58. An elastic particle is let fall from a given height above a smooth inclined plane, prove that the interval occupied by any given number of loops is independent of the inclination of the plane.

59. It is observed that a ball striking a block of wood with velocity v penetrates m feet; prove that in passing through a board n feet thick (the resistance being uniform and the same as before) it would lose the velocity

$$v\left\{1-\sqrt{\left(\frac{m-n}{m}\right)}\right\}.$$

60. A perfectly elastic ball is thrown down a smooth circular well; prove that the intervals between the instants of its impact with the side of the well are all equal.

61. Prove that the interval occupied by a projectile in passing from one extremity of a focal chord to the other is equal to that in which a heavy particle falls from rest a vertical distance equal to the length of the chord.

62. Prove that, if a, α are the numerical values of any the same acceleration referred to units of time and distance t, s; τ, σ respectively $$a=\frac{s}{\sigma}\left(\frac{\tau}{t}\right)^2\alpha.$$

63. Shew that when a number of heavy particles are projected simultaneously from the same point in directions which are all in one plane, the particles at any subsequent instant will be all in a parallel plane.

64. If the acceleration due to gravity be numerically represented by 9600 and a yard is the unit distance, shew that the unit of time is half a minute.

65. When the wind is blowing parallel to a line of railway on which are two trains having equal velocities in opposite directions it is observed that the steam track of one is twice as long as the other; prove that the velocity of each train is three times that of the wind.

66. A mass m lbs. is placed on a smooth inclined plane of inclination 30° and the plane is made to descend vertically with 10 celos, prove that the mass will slide down the plane with 11 celos.

67. AB is the vertical diameter of a circle; through A the highest point a chord AC is drawn, and the tangent to the circle at C meets the tangent at B in the point T, shew that the time of sliding down AC varies inversely as AC.

68. If two vertical circles have a common highest point, then if any line be drawn through that point the time of descending the part of the line intercepted between the circle is constant.

69. If two vertical circles touch each other at their common lowest point and a straight line be drawn from that point to cut the inner and meet the outer circle, shew that the time of sliding down the part of the line intercepted between the circles is constant.

70. Two equal masses connected by a string are placed upon two smooth inclined planes having a common altitude and inclinations α and β; prove that the acceleration of their centre of gravity is $$g \sin \tfrac{1}{2}(\alpha - \beta) \cos^2 \tfrac{1}{2}(\alpha + \beta).$$

71. A series of particles starting simultaneously from the vertex slide down the smooth sides of a pyramid; shew that after the interval t they are all on the surface of a sphere whose radius is $\frac{1}{4}gt^2$.

72. A weight W is connected with a weight ω by a system of n moveable pulleys in which each string passes round one pulley and has one extremity fastened to a fixed point; shew that the acceleration of W upwards (neglecting the weight of the pulleys) is $$\frac{2^n\omega - W}{2^{2n}\omega + W} g.$$

73. If in a system of pulleys P descends 1 foot when W ascends n feet, prove that the upward acceleration of a mass W due to a tension P is
$$\frac{nP-W}{W}g.$$

74. If in the above system P be a mass under the action of gravity, prove that the acceleration of W is
$$\frac{nP-W}{n^2P+W}g.$$

75. In a second system of pulleys in which the weight W is observed to descend 1 foot while P ascends n feet, P and W are each a mass under the action of gravity and it is observed that at a certain instant W is descending with the velocity 1 foot per second, and that after 1 second it is descending with the velocity 5 feet per second; find the relation between P and W.

76. A smooth small pulley is placed at the common vertex of two smooth inclined planes of angles α, β respectively; a string passing over the pulley connects two masses m_1, m_2 which are sliding on the planes under the action of gravity their motion being in one vertical plane. Prove that their centre of gravity describes a straight line with the acceleration
$$\frac{2g(m_1 \sin\alpha - m_2 \sin\beta)}{(m_1+m_2)^2}\sqrt{\{m_1^2+m_2^2+2m_1m_2\cos(\alpha+\beta)\}}.$$

77. Two perfectly elastic balls are simultaneously dropped from two points not in the same horizontal line, shew that their centre of gravity will never again attain its initial height, unless the square roots of the initial heights of the balls are commensurable.

78. A railway train and its engine of 60 horse-power weighs 50 tons when empty and 100 tons when loaded. Find the maximum velocity in each case supposing the resistance is 8 lbs. weight per ton.

• **79.** Two railway trains start from the same place with velocities of 20 and 30 miles an hour respectively in directions inclined at the angle 30°; find their relative velocity.

80. The trail of smoke from a railway engine makes the horizontal angle of 60° with the line of rails when the engine is running at the rate of 10 miles an hour and the angle 30° when the train is running 20 miles an hour; shew that the wind is blowing at the rate of 10 miles an hour.

81. A man travelling westward at the rate of 4 miles an hour observes that the wind seems to blow directly from the N. and that on doubling his speed it seemed to blow from the N.W.; prove that the velocity of the wind is a little greater than $5\frac{1}{2}$ miles an hour.

82. A row of equal perfectly elastic balls are placed at rest in a straight line, the one at the end is struck directly with velocity v by an equal perfectly elastic ball; prove that ultimately the ball at the other end will have velocity v and the other balls will be at rest.

83. ABC is a fixed horizontal circular hoop on a smooth table: a ball of elasticity e, projected from A, is reflected at B and at C and returns to A; prove that the interval from A to B is e times the interval from B to C.

84. A ball falls from a given height above a smooth elastic plane; prove that the interval before the ball ceases to rebound is the same for all inclinations of the plane.

85. In a wheel and axle the weights P and W are in equilibrium; if P is replaced by ω prove that the common centre of gravity of W and ω descends with the acceleration

$$\frac{W(\omega \sim W)^2}{(\omega + W)(PW + \omega^2)}g.$$

86. In the system of pulleys in which each pulley hangs by a separate string a weight P just supports the weight W; shew that if P be removed and a weight ω substituted the centre of gravity of the system will descend with acceleration

$$\frac{W(\omega \sim P)^2}{(P^2 - \omega W)(\omega + W)}g,$$

the weight of the pulleys being neglected.

87. Two equal particles are fastened to points B and C of a string ABC which is itself fastened to a fixed point at A, and $AB = BC$; the particles are describing horizontal circles uniformly and AB makes the angle α and BC the angle β with the vertical; prove that

$$\frac{2 \tan \alpha}{\tan \beta} = \frac{2 \sin \alpha + \sin \beta}{\sin \alpha}.$$

88. A shot of m lbs. is fired from a gun of M lbs. placed on a smooth horizontal plane and elevated at an angle α. Prove that if the muzzle velocity of the shot be v the range will be

$$\frac{2v^2}{g} \times \frac{M(M+m)\tan \alpha}{M^2 + (M+m)^2 \tan^2 \alpha}.$$

89. Two masses m_1 lbs. m_2 lbs. are connected by a weightless string which passes over a smooth peg. If the peg can support only one half of the sum of the weight of m_1 and m_2, prove that the least ratio of m_1 to m_2 is $3+2\sqrt{2}$.

90. A string passes over a smooth fixed pulley carrying a weight P at one end and a light pulley at the other; over the pulley Q is hung a string carrying weights P and $2P$ at its ends respectively. Prove that the tensions of the two strings are in the ratio 2 to 1.

91. A string loaded with a series of equal heavy masses m at equal distances along it is coiled up in the hand and held close to a fixed peg to which one end of the string is fastened. The string is suddenly let go; prove that the tension of the string at the peg after the rth section of the string has become tight is mrg and the impulse caused by the tightening is $m\sqrt{2rlg}$ where l is the length of each section. Hence shew that if a uniform chain of length a and mass M is similarly treated the strain on the peg when a length h of the string becomes tight is $3\frac{h}{l}MG$.

92. Two scale pans of equal mass M are connected by a weightless string which passes over a smooth pulley. Heavy masses M_1, M_2 are then placed in the pans. Shew that the stresses between these masses and the pans during the subsequent motion are $2M_1g\dfrac{M_2+M}{M_1+M_2+2M}$

and $2M_2g\dfrac{M_1+M}{M_1+M_2+2M}$ respectively.

93. An elastic particle slides down an inclined plane of given height and then impinges a horizontal plane; shew that the elevation of the inclined plane which gives the greatest range on the horizontal plane is 45°.

94. Two masses M_1 and M_2 are connected by a fine inelastic string which passes over a smooth fixed pulley. The larger mass M_2 rests on a horizontal table; M_1 is then raised h feet vertically above its position of rest and let fall. Shew that the interval between M_2 leaving the table and returning to it again is

$$\frac{2M_1}{M_2-M_1}\sqrt{\frac{2h}{g}}.$$

95. A particle sliding down a smooth inclined plane is observed to move over two equal distances (S) in the consecutive intervals t_1 and t_2; prove that the inclination of the plane is

$$\sin^{-1}\left(\frac{2S}{gt_1t_2} \times \frac{t_1 - t_2}{t_1 + t_2}\right).$$

96. Find the lines of quickest descent in the following cases:

(i) From a given point to a given straight line.

(ii) From a given straight line to a point.

97. In an Atwood's machine if the pulley can only support one half of the sum of the two weights shew that the least acceleration possible is $\frac{1}{2}g\sqrt{2}$.

98. A series of equally rough planes of different inclination all pass through the same fixed point A. Particles are placed at points P_1, P_2 etc. on each plane respectively and being simultaneously let go reach the point A simultaneously; shew that the locus of the points P_1, P_2 ... is a right circular cone.

99. Shew that the point A on the circumference of a vertical circle such that the time of descent down the radius AC is equal to the time of descent down the chord AB where B is the lowest point is such that the angle BCA is 120°.

100. A gun of mass M is placed on a smooth horizontal plane and fires a shot of mass m in a horizontal direction with u velos relative to the gun; shew that the velocity of the shot is

$$\frac{M}{m+M}\, u \text{ velos,}$$

and of the gun

$$\frac{m}{m+M}\, u \text{ velos.}$$

101. Shew that if the unit interval is S seconds and the unit distance $32S^2$ feet, then the unit force is the weight of the unit mass.

102. A foot and a second being the units of distance and time, a cubic foot of water (1000 oz.) being the unit mass, find the unit force.

103. A stone is dropped into a well and after 4 seconds the sound of the splash is heard, find the depth of the well assuming that sound travels at the rate of 1000 feet per second.

104. A smooth straight tube inclined at the angle a to the vertical moves parallel to itself in the same vertical plane with a uniform horizontal velocity; a heavy particle is allowed to slide down it; shew that the locus of the path of the particle in space is a parabola whose axis is parallel to the tube.

105. A particle is projected horizontally with velocity v along a smooth inclined plane of angle a, find the latus rectum of its path.

106. A number of projectiles projected from the same point in the same vertical plane with different velocities have equal vertical velocities; prove that the locus of the foci of their paths is a parabola having a common tangent at the vextex with their paths.

107. Any number of projectiles projected simultaneously in the same direction are always collinear.

108. A series of particles are projected from the same point in the same vertical plane so as to describe equal parabolas; prove that the locus of the vertices of these parabolas is another parabola.

109. A series of projectiles are thrown from one given point so as to strike another not in the same horizontal plane; prove that the foci of their paths lie on a hyperbola.

110. If tangents are drawn to two parabolic paths which have a common focus from any point on their common axis the velocities at the points of contact are equal.

111. A particle slides down a smooth inclined plane, and on leaving the plane falls freely; prove that the distance of the focus of its subsequent path from the foot of the plane is equal to the vertical height through which the particle slid.

112. A particle suspended in the car of a balloon revolves in a conical pendulum at a vertical distance a below the point of suspension, the particle making n revolutions per second. Shew that the upward acceleration of the balloon, supposing it vertical and constant, is $4\pi^2n^2a-g$.

113. A curve on a railway has a radius r, prove that the difference in level of the rails in order that the pressures on the wheels may be equal when a train is going round the curve with velocity v is $\dfrac{v^2}{\sqrt{(r^2g^2+v^4)}}\cdot a$, where a is the distance between the rails.

114. A string 2 feet long is attached to a stone of 1 lb. and the stone is whirling round in a vertical plane round a fixed centre so that the string is just kept straight. The string is then cut as the stone is passing the lowest point of its path: describe the subsequent motion of the stone and find where it will strike the ground 20 ft. below the centre of the circle.

115. One stone is fastened to a fixed point by a fine weightless rigid wire 4 ft. long; another stone is fastened to a fixed point by a fine inextensible string also 4 ft. long. Each stone describes a vertical circle under gravity with the least possible velocity. Compare their velocities at the lowest point of their paths.

116. A string 4 ft. long has one extremity fastened to a fixed point and the other to a stone of 4 lbs. which moves so that the string describes a right circular cone whose axis is vertical and vertical angle 60°; find the tension of the string and the velocity of the stone.

117. A heavy particle P of m lbs. lies on a smooth horizontal plane and is fastened to a light string which after passing over a fixed smooth pulley A supports a particle Q of n lbs., so that $n > m$ under the action of gravity. P is held in its position at rest and is suddenly set free; shew that the particle P will initially slide along the plane provided

$$n^2 \sin\theta < m\,(n\cos^2\theta + m),$$

where θ is the inclination of AP to the plane.

118. A fine inelastic string is loaded with n equal particles at equal distances l from each other; the thread is stretched and placed on a smooth horizontal table perpendicular to its edge over which one particle just hangs; find the velocity when the rth particle is leaving the table. Hence shew that if a heavy string of length a be similarly placed on the table its velocity on falling off will be $\sqrt{(ag)}$.

NOTE.—The string must be supposed to pass through a smooth small tube in the shape of a quadrant placed at the edge of the table.

119. A uniform heavy chain hangs at rest over a smooth small peg; from one end of the chain one fourth of its whole length is cut off; shew that the pressure on the peg is instantaneously diminished by one third.

120. An elastic particle is projected from a point between two vertical planes, its plane of motion being perpendicular to both; prove that the series of parabolas of which the particle describes arcs have their latera recta in geometric progression.

121. Particles are projected from the same point in the same direction with different velocities; prove that their foci all lie on a certain straight line.

122. Prove that the time of passage of a projectile from the point P to the point Q on its trajectory varies as $\tan\phi - \tan\phi'$ where ϕ and ϕ' are the inclinations to the horizon of the velocities of the projectile at P and Q.

123. If two particles are describing the same parabolic trajectory the straight line joining them envelopes an equal parabola.

124. Find the direction in which a rifle must be pointed in order that the bullet may strike a body let fall at the instant of firing from a balloon at rest. Find also the point at which the bullet meets the body supposing the balloon to be 220 yards high, the angle of elevation from the position of the rifleman to be 30° and the velocity of the bullet to be 2 miles per minute.

125. A particle suspended by a string l feet long is struck horizontally so that it receives a velocity u; shew that the string will begin to slacken after the body has described an arc whose cosine is $\frac{2gl - u^2}{3gl}$ provided u^2 is $< 2gl$.

126. Shew that the particle in the last question will rise to a vertical height $\frac{(u^2 - 2gl)(5g^2l^2 + 4glu^2 + u^4)}{54g^3l^2}$ feet.

127. Supposing the perfectly flexible band of the friction-brake of an engine extends over the upper half of the fly-wheel of radius r feet, and that the band is kept tight by means of a weight of W pounds hung at one end and a spring balance at the other: prove that if the spring balance registers a tension of W' pounds when the engine is making n revolutions a minute, the horse-power of the engine is

$$2\pi nr(W - W') \div 33{,}000.$$

128. Two points move with velocities v and $2v$ respectively in opposite directions in the circumference of a circle; in what positions is their relative velocity greatest and least, and what values has it then?

129. A coach wheel rolling with uniform velocity throws off continually small particles of dust at a tangent to its circumference; find the points on the wheel from which the particles will rise to the greatest height.

130. A body of 2 lbs. is projected with 20 velos at 60° to the horizon; another body of 3 lbs. is at the same instant projected from the same point at 30° to the horizon with 40 velos; find to two places of decimals the height to which their common centre of mass rises and the point at which it passes through the horizontal plane through the point of projection. [$g = 32$ celos.]

131. Find the energy per second of a waterfall 30 yds. high and a quarter of a mile broad, where the mass of water is 20 feet deep and arrives at the fall with the velocity $7\frac{1}{2}$ miles per hour; the weight of water is 1024 oz. per cubic foot. [$g = 32$ celos.]

132. Find the locus of points from which inelastic particles let fall on a smooth inclined plane may have equal velocities at the instant of passing a given horizontal line in the plane.

133. If three bodies are projected simultaneously in the same vertical plane from the same point, prove that the area of the triangle formed by joining the three bodies at any instant will vary as the square of the time.

134. A railway engine of 9 tons passes round a curve 600 feet in radius with a velocity of 30 miles per hour, prove that the horizontal force which must be exerted by the rails so that the engine may move on this curve is the weight of 2032·8 lbs. [$g = 32$ celos.]

135. A railway train is moving at the rate of 28 miles an hour when a pistol shot strikes it horizontally in the direction making the angle $\sin^{-1}\frac{3}{5}$ with the train. The shot enters a compartment of one of the carriages at the corner farthest from the engine, and passes out at the diagonally opposite corner, the compartment being 8 feet long and 6 wide. Prove that the shot is moving at the rate of 80 miles an hour and that it traverses the carriage in $\frac{5}{44}$ of a second.

136. If balls be fired at the same instant from two cannons with equal velocities at angles of elevation α and β respectively so as both to pass through a given point, prove that if t be the interval between their passing through this point and t' the interval between their returning to the horizontal plane through the point of projection, then

$$t' = 2t\cos^2\tfrac{1}{2}(\alpha + \beta).$$

137. A ball at rest on a smooth horizontal plane at the distance of one yard from a wall is impinged on directly by another equal ball moving at right angles to the wall with the velocity of one yard in a minute; the coefficient of elasticity between the balls and between the balls and wall is $\frac{1}{2}$; prove that the balls will impinge a second time after the interval $2' \, 24''$ (the radii of the balls being inconsiderable).

138. A watch is laid face upwards on a table and is moved without rotation parallel to the line joining 9 and 3 o'clock, with the velocity which the extremity P of the minute hand moves when the watch is at rest; prove that the velocity of P relatively to the table is perpendicular and proportional to the line joining P at the instant, with the position of P at any half hour.

139. A wet umbrella is held with the handle upright and made to rotate round that handle at the rate of 14 revolutions in 33 seconds. If the rim of the umbrella is a circle of one yard diameter and its height above the horizontal ground be 4 feet, prove that the drops shaken off from the rim meet the ground in a circle 5 feet in diameter. $\pi = \frac{22}{7}$; $g = 32$ celos; the effect of the air being neglected.

140. Three bodies are projected simultaneously from the same point in the same vertical plane, one vertically upwards, the second at the angle of elevation 30°, and the third horizontally, their velocities are in the ratio 1, 1 and $\sqrt{3}$; prove that they will be always collinear.

141. Two strings pass over the same smooth pulley; on one side they are both fastened to a mass M lbs., and on the other to masses P lbs. and Q lbs respectively, shew that their tensions are respectively $\dfrac{2MPg}{P+Q+M}$ and $\dfrac{2MQg}{P+Q+M}$.

142. AB is a quadrant of a circle centre O, OB being horizontal; C is a point on the quadrant; prove that the intervals of falling from A to C, and from C to B are as

$$\sqrt{\cos \tfrac{1}{2} BOC} \text{ to } \sqrt{\sin \tfrac{1}{2} BOC}.$$

143. Prove that the acceleration with which a mass descends a rough inclined plane of angle a, the limiting angle of resistance being ϕ, is $g \sin (a - \phi) \operatorname{cosec} \phi$.

144. A smooth right-angled wedge is placed with its longest face on a horizontal table; a mass M descending vertically draws another mass m up the slant side by means of a fine string, prove that the force necessary to keep the wedge at rest is

$$mg \cos a \frac{m \sin a - M}{m + M}.$$

145. A truck of m tons is drawn from rest by a horse through a feet, and then it has v velos. If the resistances are equivalent to c lbs. weight per ton, prove that the work done by the horse is $(35mv^2+mac)$ foot-pounds.

146. A donkey walks on the inside of a rough wheel 16 feet in diameter at the rate of 6 miles an hour, and brings up a bucket of water of 30 lbs. from a well 66 feet deep by means of a rope wound round the axle of the wheel 2 ft. in diameter; shew that he works at the rate of $\frac{2}{5}$ of one horse-power.

147. Prove that the greatest load which a chain capable of bearing a dead weight of W tons can draw out of a vertical shaft h feet deep in t seconds is

$$W\left(1-\frac{h}{\frac{1}{2}gt^2}\right) \text{ tons.}$$

148. Shew that the *time* of quickest descent from a straight line to a circle in the same vertical plane is

$$\sec\frac{\theta}{2}\sqrt{\frac{2l}{g}},$$

where θ is the inclination of the line to the horizontal, l the shortest distance between it and the circle.

149. When two perfectly elastic particles P, Q impinge with velocities u and v, prove that the energy interchanged between them is

$$\frac{2PQ}{(P+Q)^2}(u-v)(Pu+Qv).$$

150. Heavy particles are projected horizontally with different velocities from the same point; shew that the latera recta of the parabolas which they severally describe lie on a cone of which the axis is vertical and the vertical angle $2\tan^{-1}2$.

151. Two perfectly elastic balls A and B impinge upon each other. First A impinges upon B at rest and goes off in a direction making the angle θ with the line joining their centres; then B impinges upon A at rest at the same angle of incidence and goes off at the angle θ'; prove that $\theta+\theta'=\pi$.

152. Three equal particles are projected, each from an angular point of a triangle along the sides taken the same way round with velocities proportional to the side along which they move, prove that their C.G. remains at rest.

153. Two points are moving each with uniform velocity in the same plane; if a be the distance between the points at any

time, V their relative velocity, and u, v the resolved parts of V in and perpendicular to the direction of a, shew that their distance when they are nearest to one another is $\frac{av}{V}$, and the interval after which they arrive at this nearest distance is $\frac{au}{V^2}$.

154. A particle is projected with elevation α from a point on an inclined plane of angle β; prove that if γ is the angle of elevation of the path at the point most distant from the plane

$$\tan\alpha + \tan\beta = 2\tan\gamma.$$

155. A bird of mass M is flying horizontally at the height h with velocity v, when it is struck by a bullet of mass m moving vertically upwards with velocity V; the bullet kills the bird and remains embedded in it; prove that (neglecting the resistance of the air) the bird will fall to the ground at the distance

$$\frac{Mv}{(M+m)^2 \cdot g} \cdot [mV + v\,\{m^2v^2 + 2hg(M+m)\}]$$

from the point from which the bullet was fired.

156. Prove that the least velocity with which a particle can be projected from one point so as to pass another at the distance d and the vertical height h above it is

$$\sqrt{g(h+d)};$$

prove also that if the velocity u be greater than this, the angle between the two possible directions of projection is

$$\tan^{-1}\frac{\sqrt{(d^2-h^2)}\,\sqrt{\{(u^2-gh)^2-g^2d^2\}}}{g(d^2-h^2)+u^2h}.$$

157. From any point A on a given horizontal plane a particle slides down a smooth straight tube terminated at a fixed point B and then moves freely; shew that for different positions of A, the maximum range on another given horizontal plane situate u below B is $2\sqrt{(hk)}$, where h is the vertical depth of B below A.

158. A perfectly elastic particle is projected with velocity v at the elevation 60°; a smooth inclined plane of angle 30° passes through the point of projection; prove that if the particle returns to the point of projection it will do so after the interval

$$\frac{2v\sqrt{3}}{g}.$$

159. A particle is projected with velocity v from the foot of an inclined plane of angle a so as to strike the plane at right angles; prove that its range on the plane is

$$\frac{2v^2}{g} \cdot \frac{\sin a}{1+3\sin^2 a}.$$

160. A perfectly elastic particle strikes in succession all the fixed sides of a polygon inscribed in a circle and describes the same path continuously; prove that its path between two consecutive impacts is parallel to the tangent to the circle at the point of intersection of the sides where the impact takes place.

161. Heavy particles are projected simultaneously from points on the circumference of a vertical circle, with the velocity each would acquire in sliding from the highest point down the chord to that point, and in the direction towards the highest point. Prove that they all reach the circumference again simultaneously, and that after an equal interval they will be as the circumference of a circle of three times the radius.

162. Three elastic spheres A, B, C are at rest on a straight line; A impinges with given velocity on B, and B subsequently on C: prove that A and B will have a second impact when

$$\frac{e^2+e+1}{e}$$

is greater than $$\frac{B}{AC}(A+B+C),$$

where e is the coefficient of restitution.

163. Two balls are moving in the same straight line, one of them only being under the action of a force; the force is constant and tends towards the other ball: prove that the intervals between successive impacts decrease in Geometrical Progression.

164. A ball of elasticity e is projected from a point on a fixed inclined plane, and after once impinging upon the plane, rebounds to the point of projection; prove that, a being the inclination of the inclined plane to the horizon, and β that of the direction of projection to the inclined plane

$$\cot a \cdot \cot \beta = 1 + e.$$

165. Three balls A, B, C whose masses are as $17:1:4$ lie on a horizontal plane in a straight line; A is projected to strike B which then strikes C and is struck by A again. The coefficient of elasticity being $\frac{1}{2}$, shew that the velocities of B after its two collisions with A are the same.

166. Two perfectly elastic particles are projected at the same instant from a point on a smooth horizontal plane; prove that their C. G. will describe a number of arcs of the same parabola in different positions.

167. A weight hangs by an inextensible string of length l; shew that if the upper end of the string is suddenly made to describe a circle of radius a in a vertical plane with angular velocity ω, the string will initially not become slack unless

$$a(l-a)\omega^2 \text{ exceeds } lg.$$

168. A cannon ball of mass m is shot from a gun of mass M which is so mounted as to be free to recoil in a horizontal direction, so that its muzzle velocity, when fired horizontally, relative to the ground is V. Prove that the greatest range is $\frac{V^2}{g}$ and that this is obtained by giving the gun the elevation

$$\cot^{-1}\left(1+\frac{m}{M}\right).$$

169. Two particles are connected by a string which passes through a smooth small ring attached to a pivot; each particle describes a horizontal circle; prove that if the times of rotation are the same the paths lie on the same plane.

170. A gun is fired from a moving platform, and the ranges of the *shot* are observed to be R and S when the platform is moving forwards and backwards respectively with velocity V: prove that the elevation of the gun is

$$\tan^{-1}\left[\frac{g}{4V^2}\,\frac{(R-S)^2}{R+S}\right].$$

171. A string passing over a smooth fixed pulley supports at its ends two smooth moveable pulleys of mass P and Q respectively. Over each of these passes one of two strings each having masses P and Q at its ends. Prove that the acceleration of each of the moveable pulleys is

$$\frac{P^2-Q^2}{P^2+10PQ+Q^2}\cdot g.$$

172. A string passing over a smooth fixed pulley supports at one end a mass m, at the other a smooth small moveable pulley over which passes a string supporting masses m_1, m_2

at its ends. If T, T' be the tensions of the strings respectively, prove that

$$\frac{2T}{m} + T'\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = 4g.$$

173. A mass M rests on a smooth horizontal table; to it is attached a fine string which passes over a fixed pulley on the table, and has a pulley of mass m hanging at the other end; over this pulley passes a second string which hangs vertically and has masses m, m' attached to its ends; prove that the tension is

$$Mg\left\{1 - \frac{M(m+m')}{(M+m)(m+m')+4mm'}\right\}.$$

174. A light string has masses m_1, m_2 fastened to its ends. The middle part of the string lies stretched on a smooth horizontal rectangular table, and is perpendicular to two opposite edges of the table over which its ends hang. A particle m is knotted on the part of the string which lies on the table. Shew that the tensions of the two parts of the string are in the ratio

$$\frac{1}{m_2} + \frac{2}{m} : \frac{1}{m_1} + \frac{2}{m}.$$

175. A train stopping at two stations two miles apart take four minutes on the journey. Assuming that its motion is first that of uniform acceleration α and then of uniform retardation β, prove that (a mile and a minute being the unit distance and unit interval)

$$\frac{1}{x} + \frac{1}{y} = 4.$$

176. Two particles are projected from the same point at the same instant in the same vertical plane, with velocities u and v and in directions inclined α and β respectively to the horizon; prove that after the interval

$$t \equiv uv \sin(\beta - \alpha)/g(u\cos\alpha - v\cos\beta),$$

they will be moving in the same straight line at the distance

$$t\sqrt{\{u^2 + v^2 - 2uv\cos(\alpha - \beta)\}}.$$

177. A particle of weight W is fastened to the middle point C of a string ACB two yards in length, the ends of which, A and B, are fastened to fixed points in the same horizontal line at such a distance from each other that when the string hangs at rest each portion of the string is inclined 60° to the vertical.

The particle when hanging at rest receives an impulse perpendicular to the plane ACB, which imparts to it the velocity 16 feet per second; prove that it will describe a vertical circle completely and that the greatest and least tensions of the string are respectively $\frac{19}{3}W$ and $\frac{1}{3}W$.

178. Corn falls uniformly through an opening to a floor at the depth h below; prove that the centre of gravity of the falling corn is at the distance $\frac{1}{3}h$ from the opening.

179. A point moves with acceleration f for the interval t, with acceleration $2f$ for the next interval t, with acceleration $3f$ for the next interval t and so on. Shew that the distance described in the interval nt is $\frac{1}{12}n(n+1)(2n+1)ft^2$.

180. A body which is known to be moving with uniform acceleration is observed at a certain time to have a velocity of 880 feet per second, and four seconds after its velocity is observed to be 924 feet per second. The measure of the acceleration is 72000 and the measure of the first velocity is 600; what are the units of length and time?

181. Shew that the difference in the apparent weight of a pound carried in a train moving at the rate of 60 miles an hour, first West then East, along the circle of latitude $\cos^{-1}\frac{9}{10}$ is about 5 grains.

182. A particle A is projected towards a certain point P with a given velocity; at the same instant an equal particle B is dropped from P; prove that they meet: prove also that if they coalesce the latus rectum of their new path is one quarter that of the original path of A. If P be on the tangent at the vertex of the original path of A, prove that the point of collision is the vertex of the new parabolic path.

183. A mass M is moving on a smooth horizontal table with uniform velocity in a circle, being attached by an inextensible string to a fixed point. It hits another particle of mass m, lying at rest, which sticks to it. The tension of the string just before collision is T and just after collision it is T', shew that $(m+M)T'=MT$.

184. The sides of a rectangular billiard table are of lengths a and b. A ball of elasticity e is projected from a point in one of the sides of length b to strike all four sides in succession and continually retrace its path, shew that the angle of projection (θ) with the side is given by $ae\cot\theta=c+ec'$, where c and c' are the parts into which the side is divided by the point of projection.

185. The masses P and W hang in equilibrium under gravity over an ordinary wheel and axle; if P is pulled downwards by a uniform force R, prove that neglecting the mass of the wheel and axle the upward acceleration of W is $\dfrac{R}{P+W}$. If instead of the force R, a mass R were attached to P, then the upward acceleration of W would be $\dfrac{Rg}{P+W(1+R/P)}$.

186. At the same instant a perfectly elastic ball is dropped from A, and another equal perfectly elastic ball is projected vertically from B a point vertically below A. The two particles after impact arrive at their original starting points simultaneously; prove that the velocity of projection of B was $\sqrt{(2g \cdot AB)}$. Prove that it is impossible for the equal balls to arrive at their original starting point simultaneously unless they are perfectly elastic.

187. A particle is projected from a point in a horizontal plane and when it reaches its highest point it meets a particle of equal mass which has fallen through a vertical distance equal to its greatest height and coalesces with it. Shew that the whole range on the plane is now $\frac{1}{8}(3+\sqrt{5})$ of the free range.

188. Two masses M, m, of which m is a smooth pulley, are connected by a thread, M rests on a smooth horizontal table and m hangs vertically; m carries another thread having masses m_1, m_2 tied to its ends; prove that the acceleration of M is

$$g\left\{\frac{m(m_1+m_2)+4m_1m_2}{(M+m)(m_1+m_2)+4m_1m_2}\right\}.$$

189. If the earth were a homogeneous sphere rotating so fast that bodies at the equator had no weight then in any latitude the plumb-line would point to the pole star.

190. Two particles are projected in different directions with equal velocities from the same point A to strike the same point B; prove that if T_1 is the time taken by the first particle from A to the highest point of its path and t_1 the time from that point to B, and T_2 and t_2 similar quantities for the other particle, then $T_1^2-t_1^2=T_2^2-t_2^2$.

191. A fine inextensible string passes over two smooth fixed pulleys and has masses P and Q attached to its extremities; the string hangs down between the pulleys and another smooth pulley of mass R hangs freely in the loop. Prove that the

tension of the string is $4g\dfrac{PQR}{4PQ+(P+Q)R}$, the part of the string being vertical.

192. A light inextensible string has one extremity fixed, a heavy particle attached at the other extremity and another heavy particle attached at an intermediate point; the particles describe horizontal circles with the same angular velocity ω. Prove that the inclinations of the two parts of the string to the vertical are $\tan^{-1}g/\omega^2 r_1$ and $\tan^{-1}g/\omega_2 r_2$, where r_1 and r_2 are the radii of the circles described by the centre of gravity of the two particles and by the lower particle respectively.

193. An inclined plane of mass M and inclination a is capable of motion on a smooth table; a smooth spherical inelastic particle of mass m is projected with velocity V along the table, in a vertical plane with the line of greatest slope on the inclined plane; shew that (if the inclined plane be high enough) the particle will move along it through the vertical height

$$\frac{V^2}{2g}\frac{M^2\cos^2 a}{(M+m)(M+m\sin^2 a)}.$$

194. A string passes over two fixed pulleys and has masses m_1 and m_2 at its extremities. Between the fixed pulleys it passes through a moveable pulley of mass $\dfrac{4m_1m_2}{m_1+m_2}$; prove that the moveable pulley, if at rest, will remain at rest.

195. Prove that the interval of time in which it is possible to cross a road of breadth c in a straight line, with the least velocity, between a stream of omnibuses of breadth b, following each other at intervals of space a moving with velocity V is

$$\frac{c}{V}\left(\frac{a}{b}+\frac{b}{a}\right).$$

196. If the velocity v at any point of the path of a projectile is suddenly diminished by one half, prove that the focus of the new trajectory is instantaneously shifted nearer the projectile through the distance $3v^2/8g$.

197. Two smooth inelastic spherical balls equal in all respects are in contact with each other on a smooth horizontal plane and a third equal ball is let fall upon them so that their centre descends in their vertical common tangent. Prove that the descending ball loses six-sevenths of its velocity by the impact.

198. A smooth particle of weight W is free to move in a fine straight tube which is inclined to the horizon at the angle a and is constrained to move upward with a constant vertical velocity V; prove that the pressure on the tube is $W \cos a$.

199. Prove that when a train is running round a curve of radius r with velocity v the weight of a carriage is divided between the outer and inner rail in the ratio $gra + v^2h$ to $gra - v^2h$, where h is the height of the centre of mass of the carriage above the rails and $2a$ the distance between them.

200. A train weighs M tons; the frictional resistance is p pounds per ton; when the brake is put on there is an additional total resistance of R pounds. The engine at starting exerts a uniform pull of P pounds till the speed is v miles per hour; which speed is then maintained uniform till it is necessary to shut off the steam and put on the brake to pull up. Shew that if t be the number of hours occupied in running between two stations s miles apart then

$$vt - s = \tfrac{77}{5400} \frac{v^2 M (R+P)}{(P - Mp)(R + Mp)}.$$

201. From a point in an inclined plane of angle a is projected a particle which afterwards strikes the plane and rebounds vertically; prove that the angle which the direction of projection makes with the plane is $\cot^{-1}\{2 + e \tan a\}$, where e is the coefficient of friction.

202. A goods train consists of a number of similar wagons and an engine whose mass is an integral number, μ times that of a wagon. They are coupled together by chains of equal lengths, inelastic, and without weight. The train is initially at rest on a straight line of railway, with all its vehicles in contact and the couplings slack. The engine then begins to move, the steam exerting a constant tractive force, and each wagon is started with a jerk as its coupling tightens. Shew that during the starting the velocity of the moving part of the train will be greatest just before the n^{th} impact where $n = 3\mu^2 - 3\mu + 1$, provided there are at least n wagons, neglecting rotation and friction etc. of wheels.

203. At the points of trisection P and Q of a weightless string $APQB$ of length $3c$, equal heavy particles are fastened and the system is suspended by its ends from two fixed points A and B in the same horizontal line at the distance $2c$ from each other. The string BQ is then suddenly cut through; prove

that the tension of AP is instantaneously changed in the ratio of 6 to 7, and that the initial direction of motion of Q is inclined to the vertical at the angle $\tan^{-1}\frac{\sqrt{3}}{7}$.

204. Two equal particles lie on a smooth horizontal table connected by a straight string; if one of them is projected at right angles to the string prove that the initial radius of curvature of its path is double the length of the string.

205. Two balls of equal mass of elasticity e are projected from the same point at the height h above a fixed horizontal plane. One ball is projected vertically upwards with velocity V, the other vertically downwards with velocity U; the latter ball hits the plane and rebounds; when the balls collide they are moving with equal speeds. Prove that

$$e^2U^2 = V^2 + 2g(1 - e^2)h.$$

Also prove that the velocity of each after collision is

$$\tfrac{1}{2}e(1+e)\sqrt{(2gh + U^2)} - \tfrac{1}{2}e(U + V).$$

206. Two pulleys of masses M, M' are connected by a fine string passing over a fixed pulley. Masses m_1 and m_2 are hung over M by a fine string, and masses m_1' and m_2' are hung over M'; shew that the acceleration of M is

$$g(M + 2\mu - M' - 2\mu') \div (M + M' + \mu + \mu'),$$

where μ is the harmonic mean between m_1 and m_2, and μ' of m_1' and m_2'.

207. Shew that when a uniform heavy chain revolves with uniform angular velocity in a horizontal plane about one end which is fixed, the chain being always straight, the tension at its middle point is three-quarters of the tension at the fixed end.

208. At the points of trisection C and D of a weightless string $ACDB$ of length $3l$, equal heavy particles are fastened and the system is suspended by its ends from two fixed points at the distance $l + 2l\sin a$ from each other; the string BD is cut; prove that the tension of AC is instantaneously changed in the ratio $2\cos^2 a : 1 + \cos^2 a$.

209. A certain physical quantity is represented algebraically by a single term. When the unit of length is doubled the measure is a quarter of its former value; when the units of time and length are each doubled the measure is doubled also. If in the

first case the unit mass is defined as the mass of unit volume of a given substance, the measure is then $\frac{1}{32}$ of its original value. What kind of quantity is it?

210. A sphere is projected at the angle 30° to the horizon, and at the same instant another sphere of half the weight is projected at the angle 60° to the horizon; the centres of the spheres are in the same horizontal line and the velocities of projection are such that the centres of the spheres are always in the same horizontal line; the modulus of elasticity is $\frac{1}{2}$; shew that the paths of the spheres after impact are parabolas whose latera recta are as 4 to 9.

EXAMINATION PAPERS.

I. Cambridge Previous Examination. Part III. Additional Subjects. *October*, 1886.

(Questions **1** and **2** Trigonometry.)

3. Explain what is meant by uniform acceleration.

A stone is thrown vertically upwards with a velocity of 36 feet per second. After what times will its velocity be 12 feet per second?

4. A particle starts from rest and moves for t seconds with an acceleration f feet per second per second. Prove that the space described is $\frac{1}{2}ft^2$ feet.

Find the space described by a falling stone in the eleventh second of its fall.

5. A railway train moving with velocity 48 miles an hour has its velocity reduced to 16 miles an hour in five minutes. Find the space passed over in the interval, the retardation being assumed to be uniform.

6. Enunciate and illustrate Newton's first and second laws of motion.

A certain force acting on a particle of mass M produces an acceleration f. Shew that, if suitable units be chosen, Mf may be taken as the measure of the force.

7. A bullet is projected at an inclination θ to the horizon ($\cos\theta = \frac{4}{5}$) with a velocity of 1200 feet per second. Find the greatest height it attains and its range on a horizontal plane through the starting point.

8. Find the time that a particle takes to slide from rest down a smooth plane of length a which is inclined at an angle α to the horizon.

Shew that the times of descent of a particle down all chords drawn through the highest point of a vertical circle are equal.

II. Cambridge Previous Examination. Additional Subjects. *December*, 1886.

Question **1.** Trigonometry.

2. How is acceleration measured when variable?

Prove the formula for uniform acceleration $v^2 = 2fs$.

3. Enunciate and prove the Parallelogram of Velocities.

A stone is dropped from a height of 4 feet above the floor of a railway carriage by a person in the carriage. Find its velocity on striking the floor if the train is travelling at 15 miles an hour.

4. A ball is thrown vertically upwards with velocity g, and one second later another ball is thrown up from the same point with velocity $2g$. When and where will it strike the first ball?

5. Shew from the second law of motion that if a body be acted upon by a given force the acceleration produced varies inversely as the mass of the body.

The forces acting upon a train have a resultant in the direction of motion equal to the weight of 1155 lbs. Find the mass of the train if it acquires a velocity of 60 miles an hour from rest in 8 minutes.

6. Find the time of sliding from rest down a length a of a smooth plane inclined at an angle α to the horizon.

If the times of sliding from rest at A down two straight lines AB, AC are proportional to their lengths, shew that BC is horizontal.

7. Masses M and m hang at the ends of a string which passes over a smooth pulley. Find the acceleration.

If M strike the ground one second after the commencement of the motion, how much longer will it be before m first comes to rest?

8. Prove that the greatest height attained by a projectile above the horizontal plane through the point of projection is equal to $\dfrac{v^2}{2g}$, where v is the vertical component of the velocity of projection.

If the greatest height is equal to the range, find the tangent of the angle of projection.

III. Cambridge Previous Examination.

Additional Subjects. *June*, 1887.

1. Define velocity. How is it measured? Express approximately in yards per minute the velocity of a point on the Earth's equator, the radius of the Earth being 4000 miles.

2. Enunciate the law of composition of velocities. If a particle be simultaneously animated with velocities u and $2u$ in directions inclined to each other at 120^0, find the direction of the resultant velocity.

3. Define uniform acceleration. Express an acceleration of one foot per second per second in miles and hours.

4. If a particle under uniform acceleration f describe a space s in the time t, prove that $s = \frac{1}{2}ft^2$.

A train which is travelling at the rate of 40 miles per hour is stopped by a uniform retardation in half a mile. What is the time taken in stopping?

5. A particle is projected vertically upwards with a velocity of 40 feet per second: to what height will it rise, and what height will it attain after the expiration of half the time of reaching its greatest height?

6. Enunciate Newton's First Law of Motion, and give facts in illustration of it.

7. Prove that the range of a projectile is proportional to the product of its horizontal and vertical velocities of projection.

Hence shew that if the velocity of projection be given the range is greatest when its horizontal and vertical components are equal.

IV. Cambridge Previous Examination.

Additional Subjects. *October*, 1887.

1. A body moving uniformly passes over a distance of s feet in t seconds; find the velocity of the body.

A man walks, at the rate of 3 miles an hour, in a direct line away from a lamp which is 11 feet above a level path. Find the velocity of the shadow cast upon the path by his hand, which he keeps close to his side, at a height of $3\frac{1}{2}$ feet from the ground.

2. Two trains start at the same time from the same station and move along straight lines of railway, which are at right angles to each other, at the rates of 25 and 60 miles an hour respectively. Find their relative velocity.

3. A body starting with velocity u, and moving with uniform retardation f, has a velocity v after it has passed over a distance s; prove that $v^2 = u^2 - 2fs$.

A train, which was going at the rate of 60 miles an hour, has a uniform retardation applied to it, which reduces its velocity to 30 miles an hour, while the train goes a mile and a half. In how many minutes is the train brought to rest?

4. How is the measure of an acceleration altered if the unit of time be changed from a *second* to a *minute?*

What will be the measure of the acceleration due to gravity if a foot and a minute be taken as units of space and time respectively?

5. Enunciate the Second Law of Motion, and prove by means of it that, if two bodies at rest receive two equal blows, the velocities produced will be inversely proportional to the masses of the bodies.

A string, which passes over a smooth peg, has one end attached to a weight of 5 lbs. which lies on the ground and a weight of 3 lbs. suspended at its other end. The smaller weight is raised vertically to a height of one foot above its position of rest and is then dropped. With what velocity will the greater weight begin to rise?

6. A particle is projected from a point in a horizontal plane with horizontal velocity u and vertical velocity v; find the greatest height which it will reach.

A bullet is fired from a gun with horizontal and vertical velocities which are respectively 275 and 96 feet per second. After $7\frac{1}{2}$ seconds the sound is heard which it makes by hitting a target, in the horizontal plane through the point of projection. Find the velocity of sound.

V. Cambridge Previous Examination.

Additional Subjects. *December*, 1887.

1. When is the velocity of a body said to be variable?

Express in feet per second the velocity of a train which is travelling at the rate of forty-five miles an hour.

2. If a particle has at the same instant two different velocities in different directions, shew how the resultant velocity may be found.

If these velocities be respectively 15 miles an hour and 22 feet per second, and if their directions be inclined to each other at an angle of 120^0; find the resultant velocity.

3. A particle starts from rest and moves with uniform acceleration f through a space s, find the time during which it has been in motion.

A stone is dropped from a balloon which is moving horizontally; how many feet will it fall in the *third* and how many in the *seventh* second of its motion? If it takes ten seconds to reach the ground, what was the height of the balloon when the stone left it?

4. How is the measure of an acceleration changed if the unit of length be changed from a yard to a foot?

5. A ball is projected vertically upwards with a velocity of 100 feet per second: after how many seconds will it be at a height of 156 feet above the ground, and what will its velocity then be?

6. Enunciate Newton's Laws of Motion, and shew that the third of these laws is in accordance with facts.

7. Find the range upon a horizontal plane of a ball projected with velocity u in a direction making an angle α with the horizon.

Two balls start from the same point in directions which are inclined to the horizon at angles of 60^0 and 30^0 respectively. If they attain the same height, find the ratio of their initial velocities.

VI. Oxford Local Examinations. Junior Candidates.

July, 1887.

1. Define velocity and acceleration.

A train moving from rest with a constant acceleration attains a velocity of 20 miles an hour in one minute. Find its acceleration, adopting the usual units of space and time, viz. a foot and a second respectively.

Find also the distance passed over by the train in the minute considered.

2. What do you mean by the resolved part of a velocity in a given direction?

A perfectly inelastic particle is let fall into a tube, the inside of which is perfectly smooth. The upper part of the tube is 2 feet long and vertical; the lower part $\sqrt{2}$ feet long and inclined at 45^0 to the horizon. Prove that the velocities of the particle at the moment of reaching the angle and at that of reaching the bottom of the tube are equal.

3. Two balls of given masses and of given mutual elasticity impinge directly while moving towards one another with given velocities. Determine their velocities after their impact and separation.

VII. Oxford Local Examinations. Senior Candidates.

July, 1887.

1. If the inch, the second, and the ounce are taken as units of length, time, and mass, what will be the units of acceleration and of force?

Distinguish between weight and mass, and obtain in terms of the above unit of force the force which is equal to the weight of one pound.

2. If a particle of mass m is projected with velocity u and is acted on by a constant force mf in the direction of motion, shew that the space passed over in the time t will be $ut + \frac{1}{2}ft^2$.

If a weight of 10 lbs. is placed on a plane which is made to descend with a uniform acceleration of 10 feet per second per second, find the pressure on the plane.

3. Find the range of a projectile on a horizontal plane passing through the point of projection, the initial velocity and direction being known.

A shell is projected in a given direction with a given velocity and, when at its highest point, explodes into two equal parts, one of which rises vertically to double its height at the moment of explosion. Where will the other part strike the ground?

4. Explain what is meant by the coefficient (or modulus) of elasticity in the theory of the collision of bodies.

A sphere of mass m impinges obliquely on a sphere of mass m' which is at rest. Shew that, if the modulus of elasticity is $\frac{m}{m'}$, they will fly off in directions which are at right angles to each other.

VIII. Cambridge Local Examination. Senior Candidates. *December*, 1886.

1. A point, starting with a velocity u, moves with an uniform retardation f; prove that, if v be its velocity after it has moved through a space s, $v^2 = u^2 - 2fs$.

2. Prove that the time of descent down a smooth inclined plane of given height varies as the secant of the inclination of the plane to the vertical.

Find the inclination of an inclined plane down which a smooth particle slides through a vertical height of one foot in half a second.

3. Prove that the path of a projectile in a vacuum is a parabola; and that the velocity at any point of the path is that acquired by falling from the directrix.

4. Two imperfectly elastic balls, of masses M, M', moving in the same straight line with velocities v, v', impinge directly on one another; determine their velocities after impact.

Find the angle at which an imperfectly elastic ball must strike a hard plane, in order that its direction of motion after impact may be at right angles to its direction before impact.

IX. First M.B. Examination. Cambridge, *June*, 1886.

1. Define the terms *velocity*, *momentum*, *acceleration*, *force*, and state the relation between the force producing motion in a body and the motion produced.

A body falling freely acquires at the end of one second a velocity of 32 feet per second. Masses of 1 and 3 lbs. hang from the two ends of a fine string suspended over a smooth pulley. At what rate will they be moving at the end of one second after they are set free?

2. Two heavy bodies are dropped at the same time, one from a height of 25 and the other from a height of 50 feet, find the height and velocity of the second when the first touches the ground.

3. What do you understand by work and energy? How much energy has a mass weighing 1 cwt. and moving at the rate of 100 yards per second?

In what units is your answer expressed?

4. Explain carefully how to find the resultant of two forces acting at a point.

A body weighing 4 lb. at rest on a smooth table is acted upon by forces of 3 and 4 lb. weight in directions oblique to the table and at right angles to each other. Shew by a diagram how to find the directions of the forces.

5. What do you understand by the mechanic advantage of a machine?

In a certain system of pulleys it is found that the power descends 1 foot, while the weight rises 1 inch. What power will be required to raise a weight of 1 cwt.?

X. OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION FOR HIGHER CERTIFICATES, 1885.

7. State and prove the parallelogram of velocities.

From the window of a train which is moving at the rate of 60 miles an hour, a stone, thrown at right angles to the train with a velocity of 66 feet a second, hits a tree which is 90 ft. from the train at the time it is struck. How far was the window from the tree when the stone was thrown?

8. Define acceleration; explain what is meant by inertia.

'The number of units of weight in any particle is equal to $32 \cdot 2 \times$ the number of the units of mass in the same particle.' Under what circumstances is this statement true?

Find the relation between the measure of the weight and of the mass of a particle when a mile and a minute are the units of length and of time respectively.

9. Prove that the velocity of a projectile in vacuo under the action of gravity at any point in its path is equal to that which a particle would have if it fell vertically from the directrix of the parabola to that point.

10. Two weights W_1, W_2 are connected by a string; one of them W_1 lies on a smooth inclined plane, angle a, the other W_2 hangs over the lower edge of the plane suspended by the string under the action of gravity; the two weights are in motion in a vertical plane. Find the tension of the string.

Find how far W_1 will move from rest in two seconds, when $W_1 = 12$ lbs., $W_2 = 4$ lbs., $a = 30^0$.

11. Define an impulse.

Explain what is meant by the coefficient of restitution in impact.

A gun of 40 tons projects a bolt of 2 cwt. Find how far the recoil of the gun will force it up a smooth inclined plane whose angle is 30^0, if the initial velocity of the bolt when fired is 2000 ft. per second.

XI. OXFORD AND CAMBRIDGE SCHOOLS EXAMINATION.

July, 1887.

1. Define 'acceleration,' and explain how it is measured.

A stone is thrown vertically upwards with a velocity of 50 feet per second; a second later, another stone is thrown vertically upwards with a velocity of 40 feet per second; taking the acceleration due to gravity as 32 feet per second per second, find when and where the stones will meet one another.

2. State the Second Law of Motion, and explain how a measure of force is obtained therefrom.

A force which would statically support a weight of 30 lbs. acts uniformly in a constant direction for fifteen seconds on a mass of 50 lbs.: find the space passed through by the body in that time.

3. Define 'work,' 'kinetic energy,' 'horse-power.'

What must be the horse-power of an engine that can keep a train of 20 tons, going at the rate of 50 miles an hour, on horizontal rails, when the friction is equivalent to a resistance of $\frac{1}{210}$ of the weight of the train?

4. A man throws a stone with a velocity of 50 feet per second at an inclination of 45^0 to the horizon: find at what distance from the man it will strike the ground, and also the greatest height it attains.

5. A ball of mass 3 lbs. impinges directly with a velocity of 4 feet per second on one of mass 5 lbs. at rest. After impact the second ball moves 9 times as fast as the first: find the coefficient of restitution between the balls.

XII. University of London. B.A. and B.Sc. Pass Examinations: 1886.

4. Shew how two coexistent velocities or accelerations may be compounded by the triangle of velocities and accelerations: and give illustrations.

Knowing the direction of the true wind, and the velocity and direction of the apparent wind on the ship as shewn by the vane on the mast, determine the velocity of the ship, supposing there is no lee-way.

5. Determine what would have to be the numerical value of g, the acceleration of gravity, in order that a body starting from rest should fall 10,000 feet in 10 seconds.

When $g=32$, determine—

(i) The distance fallen from rest in 10 seconds;

(ii) The time of falling 10,000 feet;

(iii) The initial vertical velocity in order that the body should fall 10,000 feet in 10 seconds.

6. Prove that a train going 45 miles an hour will be brought to rest in about 378 yards by the brakes, supposing them to press with two-thirds of the weight on the wheels of the engine and brake-vans, which are half the weight of the train; and supposing a coefficient of friction ·18.

7. Prove that an engine capable of exerting a uniform pull of 3 tons can take a train of 120 tons on the level from rest at one station to stop at the next station two miles off in about 3 minutes $38\frac{1}{2}$ seconds, the speed being kept uniform when it has reached 45 miles an hour, and the brakes being applied as in the last question in order to stop.

(Neglect passive resistances, and take $g=32$.)

8. Prove that a piece of mud thrown from the top of a hansom cab wheel of diameter d feet, the cab moving with velocity v feet per second, will when it strikes the ground be at a distance $\frac{1}{4}v\sqrt{d}$ in front of the position then occupied by the point of contact of the wheel with the ground.

Prove that the velocity v must exceed $4\sqrt{d}$ for the piece of mud to clear the wheel.

9. Determine the horse-power transmitted by a belt moving with a velocity of 600 feet a minute, passing round two pulleys, supposing the difference of tension of the two parts is 1650 lbs.

10. A pile is driven a feet vertically into the ground by n blows of a steam-hammer fastened to the head of the pile.

Prove that, if p is the mean pressure of the steam in lbs. per sq. inch, d the diameter of the piston in inches, l the length of the stroke in feet, w the weight in lbs. of the moving parts of the hammer, and W the weight of the pile and the fixed parts of the steam-hammer attached to it; then the mean resistance of the ground in lbs. is

$$\frac{nw}{W+w}\left(w+\tfrac{1}{4}\pi d^2 p\right)\frac{l}{a}.$$

XIII. Examinations for Scholarships at St John's College.

December, 1885.

9. Enunciate and prove the parallelogram of velocities.

A number of soldiers are marching with shouldered rifles in a shower of rain which falls vertically. Having given the velocity of the rain-drops and the rate of marching, find what must be the least inclination of the rifles to the vertical in order that no rain may enter the barrels.

10. Define acceleration, and explain how it is measured.

Taking 32 as the measure of the acceleration of a body falling under the action of gravity when a foot and a second are units of length and time, find its measure when the units are a mile and two minutes forty-five seconds.

If the acceleration of a falling body be taken as the unit of acceleration, and if the unit of velocity be the velocity it acquires in four seconds, find the units of length and time.

13. A ball moving with velocity v impinges directly on a ball at rest; this ball moves with the velocity communicated, and after a direct rebound from a wall comes again into collision with the first ball. Find the ratio of the masses if the first ball be reduced to rest.

15. Define the terms *potential energy of a system* and *kinetic energy of a system*, and find an expression for the potential energy of an elastic string when extended beyond its natural length.

Two equal particles on a smooth horizontal plane are connected by an elastic string. If the particles be drawn apart until the length of the string is double its natural length and then let go, find the greatest velocity acquired by the particles.

16. Find the direction and magnitude of the acceleration of a point which describes with given uniform velocity a circle of given radius.

If the particles in Question **15**, when drawn apart to a given distance, be projected with equal velocities in opposite directions at right angles to the string, find the velocity of projection in order that the particles may each move uniformly in the same circle. Prove that the velocities requisite for circular motion when the lengths of the string are respectively double and quadruple its natural length are in the ratio of $1 : \sqrt{6}$.

XIV. Entrance Scholarship Examination. Gonville and Caius College. *December*, 1885.

8. Define the acceleration of a moving point and state how many quantities are necessary to specify it.

The Moon's distance from the Earth is about 239,000 miles and she revolves once round the Earth in about $27\frac{1}{3}$ days. Find her acceleration relatively to the Earth with feet and seconds as units.

9. Shew that Newton's Second Law provides a direct method of comparing the magnitudes of two forces. A wedge mass M and angle α is placed on a smooth horizontal plane, and a particle mass m is placed on the wedge. Shew that the acceleration of the particle with respect to the wedge is

$$g \sin \alpha \frac{M+m}{M+m\sin^2\alpha}.$$

10. A particle is projected with given velocity under the action of gravity from a point P so as to pass through a point Q; shew that if t_1, t_2 are the two possible times of flight

$$gt_1t_2 = 2PQ.$$

11. State the experimental law on which the determination of the motion of two elastic balls after impact depends, and shew that the Kinetic Energy after impact is never greater than that before.

Three particles of equal mass are placed at the angular points A, B, C of an equilateral triangle and A, B are connected with C by equal strings which are just taut. C is projected perpendicularly towards AB. Find the loss of kinetic energy after A and B are jerked into motion.

12. When does a force do work on a body, and how is the work done measured?

Apply the methods of energy and conservation of angular momentum to the following problem. A particle which can move on a smooth horizontal table is attached to a fixed point of the table by an elastic unstretched string of length a, which is such that the weight of the particle would stretch it to twice its natural length. Prove that if the particle be projected with velocity u perpendicular to the string, its velocity when next moving at right angles to the string will be given by

$$v^3 + v^2u + vga - uga = 0.$$

XV. Entrance Scholarship Examination. Gonville and Caius College. *December*, 1887.

7. Give Newton's three laws of motion; shew from them that the measure of a uniform force is the momentum added in a unit of time to a mass on which the force acts.

Assuming a yard, a minute and a hundredweight for units of length, time and mass, express the corresponding unit force in poundals.

8. State and prove the parallelogram of velocities. Deduce from it the parallelogram of impulses.

A bird of mass M is flying horizontally at a height h with velocity V, when it is struck by a bullet of mass m moving vertically upwards with velocity v; the bullet kills the bird and remains embedded in it. Find the magnitude and direction of the impulsive stress; and prove that, neglecting the resistance of the air, the bird will fall to the ground at a distance

$$MV[mv+\sqrt{\{m^2v^2+2hg(M+m)^2\}}]/[(M+m)^2g]$$

from the point from which the bullet was projected.

10. A string on a smooth horizontal table passes through a small smooth ring of mass P; one end of it is fixed to the table, the other to a mass A lying on the table; another string, one end of which is fastened to the ring, passes over the edge of the table and has its other end fastened to another mass B, which hangs suspended by the string under the action of gravity. The system is in motion so that the path of the ring and of A and of B is each a straight line perpendicular to the edge of the table. Find the acceleration of A.

11. Two railway trucks of 10 tons each are attached by means of inelastic chains (which when the trucks are in contact have 4 feet of slack) to each other and to an engine of 40 tons. The friction between the rails and the wheels amounts to $\frac{1}{160}$ of the pressure and the inertia of the wheels etc. is neglected. To move itself and the trucks (the trucks being initially in contact with each other and with the engine) the engine can exert a continuous horizontal force equivalent to the weight of 12 cwt. Shew that the engine can move the train on a horizontal railway so that the last truck goes 1 mile in 1 hr. 32 min. nearly ($g=32$).

12. Define the hodograph of a moving point.

Find the hodograph of a point moving with constant speed on the circumference of a circle. Hence find the acceleration of such a point.

Draw the hodograph of a point moving with constant speed on the circumference of a regular polygon.

13. Define work; distinguish between power and energy.

State the principle of Virtual Work: and, assuming the principle to be true, deduce from it the triangle of forces.

XVI. College Examination.

Clare College, Gonville and Caius College, King's College.

June, 1884.

1. State Newton's laws of motion and deduce the relation between the acceleration of a mass and the force acting on it.

Two masses m, m' are attached by an inelastic string, m' rests on an inclined plane of mass M and m hangs vertically over the upper end of the plane pressing against the vertical face of the plane. If the inclined plane be free to slide parallel to a line of greatest slope along a horizontal line, find its acceleration and the pressures between it and the particles.

2. Define the following quantities, explaining how they are measured and mentioning their dimensions: Work, Power, Kinetic Energy, Potential Energy, Impulse. Justify the measure you give for Kinetic Energy.

A cannon ball weighing 10,000 grammes, is discharged with a velocity of 45,000 centimetres per second from a cannon, the length of whose barrel is 200 centimetres, prove that the mean force exerted on the ball during the explosion is $5{\cdot}0625 \times 10^{10}$ dynes.

3. A particle is projected with a given velocity in a direction making a given angle with a line of greatest slope of an inclined plane and in the same vertical plane with it; find the range on the plane and the greatest distance from the plane which the particle attains.

A particle, whose coefficient of restitution is e, is projected vertically with velocity u from a point O on a plane inclined at an angle α to the horizon; prove that the distance from O of the point where the particle meets the plane for the $(n+1)$th time is

$$\frac{2u^2}{g}\sin\alpha\,\frac{e(1-e^{n-1})(1-e^n)}{(1-e)^2},$$

and prove that the points of the different parabolic paths where their tangents are parallel to the inclined plane lie on a parabola.

4. Shew how to determine the velocity and acceleration of the centre of inertia of any given masses, knowing the velocities and accelerations of each mass; and prove that the kinetic energy of two masses is equal to the kinetic energy of the two supposed moving with the velocity of the centre of inertia, and the kinetic energy of each mass relatively to the centre of inertia.

Three masses each equal to m are attached to the extremities and middle of a rod without inertia, and of length $2a$. A blow B is applied perpendicularly to the rod at a distance x from the middle point, prove that the kinetic energy generated is

$$\frac{B^2}{6m}\left\{1+\frac{3}{2}\frac{x^2}{a^2}\right\}.$$

5. A particle of mass m revolves in a circle of radius r with angular velocity ω, prove that the force is necessarily to the centre and find its value.

Two particles of masses m, m' lie on a smooth horizontal table connected by an inelastic string of length a, m is projected in the plane at right angles to the string, prove that the initial radius of curvature of its path is $\frac{m+m'}{m}a$.

XVII. College Examination.

Clare College, Gonville and Caius College, King's College.

June, 1885.

1. Give Newton's three laws of motion and shew from the second law that the measure of a uniform force is the momentum added on per unit of time.

A particle (mass m) slides down a smooth wedge (mass M, inclination α) which is free to move on a smooth horizontal plane. Prove that the path of the particle is a straight line, and assuming the principle of energy prove that its vertical acceleration is

$$\frac{g(M+m)\tan^2\alpha}{M+(M+m)\tan^2\alpha}.$$

2. Prove the formula $s=ut+\frac{1}{2}ft^2$ for rectilinear motion with uniform acceleration.

A cylinder of height h and diameter d is standing on the horizontal seat of a railway carriage. If the train begins to move with acceleration f, prove that the cylinder will not remain undisturbed unless f is less than each of the quantities μg and $\frac{dg}{h}$ where μ is the coefficient of friction.

3. Prove that the time of sliding down all chords of a vertical circle to the lowest point is the same.

Find the position of a point on the circumference of a vertical circle, such that the time of descent from it down the radius to the centre and down a chord to the lowest point may be the same.

4. Explain the theory of Atwood's machine.

Two weights P and W balance on the wheel and axle. Prove that if they be interchanged the weight W will after one second be descending with a velocity

$$g\frac{W(W-P)}{W^2-WP+P^2},$$

the mass of the axle being neglected.

5. Prove that the path of a projectile is a parabola; and that the velocity at any point is that which would be acquired by a particle falling from the directrix to that point.

A perfectly elastic particle is let fall from a point P on the upper surface of a vertical hoop (centre O) and rebounds from the inner surface of the hoop. Prove that after two rebounds it will be moving vertically if

$$\tan\theta = 2\sin 4\theta,$$

where θ is the angle which OP makes with the vertical, and find the time which elapses before the particle arrives again at P.

6. A particle is describing a circle of radius r with velocity v, shew that the acceleration along the radius is $\dfrac{v^2}{r}$.

A string of length l has its ends fastened to two points A, B in the same vertical line, and a bead P on the string rotates uniformly about AB so that BP is always horizontal; prove that its angular velocity is equal to $l\left(\dfrac{2g}{a(l^2-a^2)}\right)^{\frac{1}{2}}$, where $AB = a$.

XVIII. College Examination.

Clare College, Gonville and Caius College, King's College.

June, 1886.

1. Give the law of motion connecting mass, force and acceleration for a moving particle. Do you consider it to be an experimental result, and if so in what respects, or definition, and if so of what?

What are the ultimate tests of equality (1) of two forces, (2) of two masses?

2. What choices as to units are commonly made in dynamics before any equations are written down?

If, other choices of units being the same as usual, we choose for unit of velocity the velocity which a particle of unit mass acquires in moving from rest through a unit length under the action of unit force, find the velocity of a body which moves through ten units of length per unit time.

3. Find the acceleration of a particle which is moving with uniform velocity in a circle.

A railway carriage is moving in a horizontal plane at the rate of 30 miles an hour round a circular curve of which the radius is 10 chains. In the centre of the carriage, and fixed with regard to it in a horizontal plane, is a smooth circular wire of 6 ft. diameter, on which a bead moves uniformly, making 20 revolutions per minute; the two circular motions being in the same direction. Find the pressure between the bead and the wire.

4. Give the arrangement and theory of Atwood's machine.

In a certain experiment with this machine, the two weights which are supposed to be equal differ by a small weight w, while their sum is correctly known, and the given distances moved through in the two stages of the motion are equal. Shew that error, due to w, in the value found for the acceleration of gravity is approximately $\frac{w}{2}\left(\frac{3}{m}+\frac{1}{M}\right)$; where m is the mass removable, and M is the whole moving mass (including the allowance for the inertia of the pulley) after the removal of m.

5. Two inelastic particles are moving in one straight line; deduce from first principles the effect of a collision between them.

A goods train consists of a number of similar wagons and an engine whose mass is an integral number, μ, times that of a wagon. They are coupled together by chains of equal lengths, inelastic and without weight. The train is initially at rest, on a straight line of railway, with all the vehicles in contact and the couplings slack. The engine then begins to move, the steam exerting a constant tractive force, and each wagon is started with a jerk as its coupling tightens. Shew that, during the starting, the velocity of the moving part of the train will be greatest just before the n^{th} impact, where $n = 2\mu^2 - 3\mu + 1$, provided that there are at least n wagons. All rotation of wheels and friction may be neglected.

6. Find the path of a projectile in a vacuum.

A solid paraboloid of revolution is fixed with its axis vertical and vertex uppermost; a particle is projected from the vertex with a given velocity, and impinges on the surface after a parabolic flight. Find the angle of projection for which the time of flight is greatest.

XIX. Mathematical Tripos, 1885.

1. Distinguish between the weight of a body and its mass. How does the weight of a body vary in different latitudes, and how would it be affected if the body were removed to the surface of the moon? Shew how the intensity of gravity at the surface of a planet fixes a superior limit to the dimensions to which its inhabitants can attain.

7. Find the range on a horizontal plane of a shot for a given angle of elevation and initial velocity; and shew that the height of the vertex in feet is approximately four times the square of the time of flight in seconds.

Prove that, projected on a vertical target, the shot, as seen from the point of projection, will appear to descend with constant velocity.

8. Find the acceleration of a body sliding up or down a rough inclined plane; and prove that the velocity at any point will be that due to falling freely under gravity from a certain straight line which slopes downwards in the direction of motion at the angle of friction.

Prove that the loss of time in going from A to C, two points on a railway at the same level 8 miles apart, due to an incline of 1 in 100 from A up to B, and an incline of 1 in 300 from B down to C, instead of going on a level line from A to C at a uniform velocity of 45 miles an hour, is about 2 min. 20 sec.

It is supposed that with full steam on the velocity drops from 45 miles an hour at A to 15 at the summit B, and that in descending the incline from B to C full steam is still kept on till the velocity has again reached 45 miles an hour, after which the velocity is kept uniform by partly shutting off steam; and prove that this happens at a point Q distant from B about 1 mile 892 yards.

9. Prove that a train of W tons going up an incline of 1 in m will acquire velocity $\left(\frac{P}{W}-\frac{1}{m}-\frac{\mu}{2240}\right)gt$, and energy $\frac{1}{2}W\left(\frac{P}{W}-\frac{1}{m}-\frac{\mu}{2240}\right)^2 gt^2$ foot-tons, in $\frac{1}{2}\left(\frac{P}{W}-\frac{1}{m}-\frac{\mu}{2240}\right)gt^2$ feet, after t seconds from rest, if P denotes the pull of the engine in tons, and μ the resistances in pounds per ton.

Prove that in Question 8 the pull of the engine from A to Q is $2\frac{5}{48}$ tons, and from Q to C is $\frac{7}{12}$ of a ton, supposing $W=200$, $\mu=14$, $g=32$; and find the extra expenditure of work due to the inclines.

10. If a body attached to its centre of mass to one end of a string of length r, the other end of which is attached to a fixed point on a smooth horizontal plane, makes n revolutions a second, the tension of the string is to the pressure on the plane as $4\pi^2n^2r$ to g.

Prove that, if a train is running round a curve of radius r with velocity v, the weight of a carriage is divided between the outer and inner rail in the ratio of $gra+v^2h$ to $gra-v^2h$, where h is the height of the centre of gravity of the carriage above the rails, and $2a$ the distance between them.

11. Calculate the velocities after direct impact of two elastic spheres.

An inelastic pile of w lb. is driven vertically a feet into the ground by n blows of a hammer of W lb., falling h feet. Prove that $\frac{nW^2}{W+w}\frac{h}{a}$ lb. superposed on the pile in addition to W would drive it down very slowly, supposing the resistance uniform.

If the pile is crushed x feet by each blow, where x is small, the mean pressure exerted by the hammer is $\frac{Ww}{W+w}\frac{h}{x}$ lb., and each blow lasts for $\frac{x}{h}$ of the time of falling of the hammer, neglecting forces not due to the impulse.

XX. MATHEMATICAL TRIPOS, 1886.

7. Define momentum, and explain how the momentum of a particle is altered by the action of a given force.

A smooth particle of weight W is free to move in a fine tube which is inclined to the horizon at angle α and is constrained to move upwards with a constant vertical velocity V; prove that the pressure on the tube is $W\cos\alpha$. Prove also that if the tube be initially at rest and the particle attached to it by a very fine thread, and if the velocity V with which the tube is started off be at least sufficient to break the thread, then at a time t after starting the possible positions of the particle in the tube cannot differ by a distance exceeding $Vt\sin\alpha$.

8. Prove the formula $s=vt+\frac{1}{2}ft^2$, and investigate the relative path of two beads free to move on two straight smooth wires in the same plane under the action of constant forces.

A telescope on a heavy platform is drawn up a smooth inclined plane of inclination α by a force so adjusted that the telescope may keep a given projectile always in the field of view. Prove that if β be the angle the telescope makes with the plane and V the initial velocity of the projectile perpendicular to the axis of the telescope, the magnitude of the force per unit mass must be $g\cos\alpha\cot\beta$ and the initial velocity of the telescope $V\operatorname{cosec}\beta$. The motion of all the bodies is supposed to be in one plane.

9. Prove that when a heavy particle is to be projected from a given point with a given velocity to pass through another given point there are two possible directions of projection.

A man standing on the edge of a cliff throws a stone with given velocity u at a given inclination in a plane perpendicular to the edge. After an interval τ he throws from the same spot another stone with given velocity v at an angle $\frac{1}{2}\pi+\theta$ with the line of discharge of the first stone and in the same plane. Find τ so that the stones may strike one another, and prove that the maximum value of τ for different values of θ is $2v^2/gw$, and occurs when $\theta=\sin^{-1}v/u$, w being v's vertical component.

10. Two elastic balls moving with given velocities strike one another directly. Find in terms of Newton's coefficient their velocities after impact and the loss of kinetic energy in the impact.

Two perfectly elastic balls of different masses are set in motion with velocities V, v within a smooth circular tube fixed in a horizontal position. If after the first impact they describe angles A_1, a_1 respectively before the second impact, and if after the second impact they describe angles A_2, a_2 respectively before the third impact, then

$$V\frac{A_1-A_2}{A_2}+v\,\frac{a_1-a_2}{a_2}=0.$$

11. Find the component, in any specified direction, of the acceleration of a particle moving with uniform velocity in a circle.

AB is a fine elastic string passing round a smooth vertical peg at a centre of force attracting with a force proportional to the distance. *AP* is a fine inextensible string carrying a series of masses, and *BQ* is another such string carrying another series. All the masses rest on a smooth horizontal plane and are set revolving about the centre with the same angular velocity, after being drawn out sufficiently far for them to continue in steady revolution with all the masses of each system in one straight line through the centre. Prove that for any two different intensities of the centre, provided the angular velocity remain the same, the difference between the total kinetic energies is proportional to the difference between the products of the momenta of the two sets of masses.

12. A heavy particle resting on the top of a smooth fixed sphere is slightly displaced. Find its pressure on the sphere at any point of its path.

Two smooth equal spheres are supported by fine equal strings attached to their surfaces and hanging from one fixed point. The spheres are now glued together at their point of contact, and a smooth cylinder is laid on them so as to rest with its axis horizontal and not to interfere with the strings. Supposing the glue suddenly to give way, find the ratio of the new instantaneous pressure of the cylinder on a sphere to the previous pressure, and prove that if each string make an angle 60^0 with the horizon and an angle 30^0 with the perpendicular from the centre of its sphere upon the axis of the cylinder, there is no change in the pressure, provided the masses of the spheres and cylinder be all equal.

ANSWERS.

I.

1. $27\frac{3}{11}$ velos. 2. 12 min. $34\frac{2}{7}$ sec.
3. (i) $27\frac{3}{11}$ miles per hour. (ii) $3\frac{9}{22}$ miles per hour.
4. (i) $\frac{11}{6300}$ velos. (ii) $\frac{11}{90720}$ velos. 5. $1536\frac{32}{63}$ velos.
6. 1 min. $1\frac{17}{43}$ sec. 7. 500 yards in $11\frac{1}{4}$ sec.; their ratio is 66 to 100.
8. 11 to 6.
9. (i) 30 velos; $20\frac{5}{11}$ miles per hour.
(ii) $26\frac{46}{49}$ velos; $18\frac{18}{49}$ miles per hour.
(iii) $20\frac{380}{421}$ velos; $14\frac{106}{421}$ miles per hour.
(iv) $17\frac{1113}{5111}$ velos; $11\frac{3779}{5111}$ miles per hour.
10. (1) $35\frac{1}{5}$ velos. (i) 24 miles per hour.
(2) $29\frac{4581}{5911}$ velos. (ii) $20\frac{1780}{5911}$ miles per hour.
(3) $32\frac{141}{360}$ velos. (iii) $22\frac{15}{176}$ miles per hour.

II.

1. $\frac{22c}{15}$ velos. 2. $\frac{15hv}{22}$ miles. 3. $\frac{3m}{k}$ seconds.
4. $\frac{45hm}{22t}$ miles. 5. $40\lambda k$ *yards*. 6. $\frac{22mt}{15k}$ hrs. 7. $\frac{22nt}{45k}$ yards.

III.

1. $10\frac{1}{2}$ velos. 2. (i) 10 velos. (ii) 11 velos. 4. 4 velos; $33\frac{1}{3}$ yds.
5. $\frac{5ab}{2b}$ velos. 6. $9v$ velos.

IV.

1. 13 velos. 2. 55 velos.
3. 255 velos, 210 velos, 165 velos, 120 velos. 4. 65 velos.
5. 277 velos. 6. 3 min. 18 sec. (after it begins to decrease.)
7. (i) 8 secs. from rest. (ii) 1 m. $10\frac{2}{3}$ sec. 8. 40 sec. previously.
9. 4 sec.; 8 sec. 10. 11 sec. 11. 96 ft.; 48 ft.; 96 ft.
12. $3\frac{1}{2}$ sec. previously; 7 sec. previously.

VI.

1. (i) 102 feet. (iii) 12,000 feet or $2\frac{3}{11}$ miles.
(ii) 208 feet. (iv) 27,600 feet or $5\frac{5}{22}$ miles.

5. 10 seconds before the 5 sec. **6.** Yes. **7.** 320 velos.
8. 81 ft. **9.** $48\frac{3}{4}$ feet. **10.** 32 ft.; $2\frac{1}{2}$ secs. previously.
11. In 3 seconds more; 54 ft. further.

VII.

1.
- (i) 80 velos.
- (ii) 176 velos.
- (iii) 320 ft.
- (iv) 6720 ft.
- (v) in $33\frac{7}{8}$ sec.
- (vi) half a sec. previously.

2. 20 celos. **4.** 640 feet. **5.** 4 celos.
6. (i) 32 celos. (ii) 144 feet.

7.
- (i) 15 velos.
- (ii) 6 sec.; 90 ft.
- (iii) $87\frac{1}{2}$ ft.
- (iv) 3 to 1.
- (v) $(6-3\sqrt{2})$ sec. $=1 \cdot 75\ldots$ secs.

8.
- (i) -32 celos.
- (ii) in $4\frac{1}{2}$ sec. from the commencement of the first interval.
- (iii) -196 ft. from its position when at rest.
- (iv) it starts with 144 velos.
- (v) in 9 sec. from commencement of first interval.

9. $\dfrac{k-2h}{3t^2}$ celos. **10.** $\dfrac{k-h}{2t+4}$.

VIII.

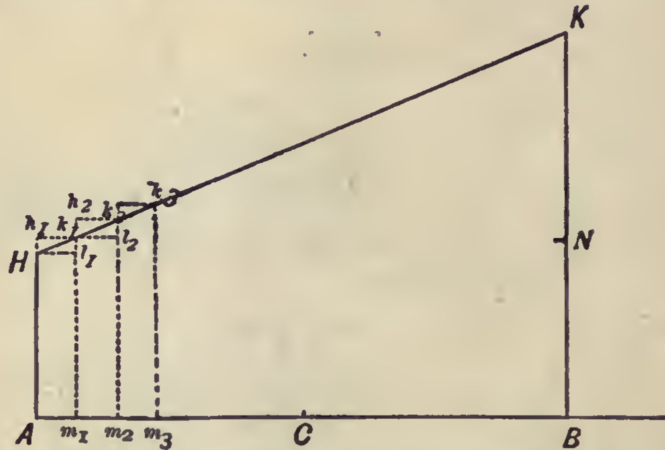
1. 3 sec. **2.** (i) $10(\sqrt{2}-1)$ sec. $=4 \cdot 14$ sec. (ii) $56\frac{1}{4}$ ft.
3. 100 sec.; 180000 ft. **4.** 40 sec.; 25600 ft.; in 20 secs.
5. 6 celos. **6.** 2694 velos (nearly). **7.** $\frac{44}{45}$ celos.
8. $-3 \cdot 3$ celos. **9.** 2640 ft. **10.** $\frac{121}{3375}$ celos.
11. It has $-\frac{5}{11}$ celos, therefore the required interval t secs. is one of the roots of the equation $300=22\frac{3}{11}t-\frac{1}{2}\times\frac{5}{11}t^2$, that is $t=49\pm 32 \cdot 88\ldots$ secs.
12. It has $-\frac{5}{48}$ celos; t is one of the roots of $400=13\frac{23}{24}t-\frac{1}{2}\times\frac{5}{48}t^2$; $t=134\pm 101 \cdot 3\ldots$ **13.** In 4 secs. from the first instant.
14. $\frac{2}{75}$ celo. **15.** 64 ft.; 64 velos. **16.** 54 celos. **17.** 1 celo.
18. 16 celos; 36 velos. **19.** Their ratio is 1 to 4 $[v^2=2as]$.
20. 256×2 ft.

IX.

1. 10 velos; 2 celos; velocity at beginning $=6$ velos; velocity at end $=14$ velos. **2.** 80 velos; 20 seconds previously; 800 ft. back.
3. $6\frac{2}{3}$ celos. **4.** 2 celos. **5.** 10 sec. **6.** 10 sec.
7. In $2\frac{1}{2}$ sec. and 5 sec. **8.** 20 celos. **10.** 4 sec.

12. 48 ft. from A. **13.** 90 feet from B.

14. In $\sqrt{\frac{a}{2g}}$ sec. at a point $\frac{1}{4}a$ feet from A. **16.** $2\frac{1}{2}$ secs.



20. In the above figure the base of each little parallelogram Am_1, m_1, m_2, etc. is τ ft.; AH is u ft.; Ah_1 is $(u+a\tau)$ ft. and so on; hence the series $$u\tau+(u+a\tau)\,\tau+(u+2a\tau)\,\tau+\text{etc.}$$
represents the sum of the areas of the little parallelograms Al_1, m_1l_2, m_2l_3, etc.

the series $$(u+a\tau)\,\tau+(u+2a\tau)\,\tau+(u+3a\tau)\,\tau+\text{etc.}$$
represents the sum of the areas of the little parallelograms Ak_1, m_1k_2, m_2k_3, etc.

The area $AHKB$ lies between the areas of these two series of parallelograms; and when the bases Am_1, m_1m_2 etc. are infinitely diminished, the area of each series of parallelograms tends to become equal to the area of $AHKB$. Therefore the area $AHKB$ represents the required distance.

Now $AB=t$ ft.; $AH=u$ ft.; $Bk=(u+at)$ ft.; the area
$$=\tfrac{1}{2}\{u+u+at\}\,t\text{ sq. ft.}=(ut+\tfrac{1}{2}at^2)\text{ sq. ft.}$$
That is, the required number of linear feet is $(ut+\frac{1}{2}at^2)$.

X.

1. $\frac{1}{2}$ celo. **2.** 32 celos. **3.** $\frac{1}{2}$ celo. **4.** $12\frac{4}{5}$ celos.
5. $\frac{3}{7}$ celo. **6.** $\frac{1}{64}$ celo. **7.** 5 velos; 25 ft.
8. 48 velos; 96 ft. **9.** 2 velos; 3 ft. **10.** $5\frac{1}{7}$ velos; $23\frac{1}{7}$ ft.
11. 3 velos; 18 ft. **12.** 16 velos; 4 ft.
13. (i) 8 velos; 18 ft. (ii) 22 velos; 84 ft. (iii) 158 velos; $283\frac{1}{2}$ ft.
14. 120 ft. **15.** 512 ft.
16. after 2 sec.; 32 velos; or, after 4 sec.; -32 velos.
17. after 2 sec.; and after 3 sec. **18.** 54 poundals.

19. 32 poundals.
20. 36 velos; 320 poundals.
21. 20 lbs.
22. $\frac{40}{11}$ lbs.
23. In the first minute it travels $6\frac{1}{4}$ ft.; it travels $14\frac{7}{9}$ ft. further before coming again to rest.
24. One is four times as great as the other.
25. $\dfrac{60np}{m}$ velos.
26. 60 poundals.
27. 640 secs.; 45 miles per hour.
28. 200 poundals.
29. In 4 secs.
30. 80 velos.

XI.

1. mg poundals.
2. $\dfrac{f t^2}{40 \times 112}$ ft.
3. $20 \times 112\, kg$.
4. $\dfrac{4928}{1125}\, mk$ secs.

XII.

1. 9 pound-velos.
2. 1120 pound-velos.
3. $100 \times 20 \times 112 \times 44$ pound-velos.
4. $62\frac{1}{2}$ pound-velos.
5. 336000 pound-velos.
6. $2956\frac{4}{5}$ pound-velos.
7. 80 pound-celos.
8. 224×32 pound-celos.
9. $\dfrac{44 \times 5 \times 20 \times 112}{3}$ pound-celos.
10. $10416\frac{2}{3}$ pound-celos.
11. $56 \times \frac{1}{20} \times (1500)^2$ pound-celos.
12. $\dfrac{88 \times 88}{2700} \times 12 \times 14$ pound-celos.

XIII.

1.
(7) 80 poundals.
(8) 32×224 poundals.
(9) $\frac{44}{60} \times 100 \times 20 \times 112$ p.
(10) $10416\frac{2}{3}$ poundals.
(11) $\frac{56}{20} \times (1500)^2$ poundals.
(12) $(\frac{88}{3})^2 \times \frac{1}{300} \times 168$ poundals.

2.
(1) $\frac{9}{20}$ poundals.
(2) 56 poundals.
(3) 492800 poundals.
(4) $3\frac{1}{8}$ poundals.
(5) 16800 poundals.
(6) $147\frac{21}{25}$ poundals.

3. $16\frac{32}{75}$ minutes.
4. 18000 poundals.
5. 80 poundals.
6. 36·5 poundals.

XIV. a.

1. 96 poundals.
2. 1 poundal.
3. $20 \times 2240 \times 32$ poundals.
4. $4 \times 112 \times 32$ poundals.
5. 4 lbs. wt.
6. $31\frac{1}{4}$ lbs. wt.
7. 100 lbs. wt.
8. 14 lbs. wt.
9. 125 lbs. wt.
10. 4×112 lbs. wt.
11. 2 celos.
12. 110 secs.
13. 480 velos.
14. $\dfrac{4 \times 100 \times 2240}{32}$ lbs. weight $=$ 28000 lbs. weight.
15. $\frac{28}{15}$ lbs. weight.
16. 14 m. 40 sec.
17. In 50 secs.
18. 308 secs.
19. 8×45 tons wt.
20. 64 lbs.
21. $(480)^2 \left\{\dfrac{1}{7680} - \dfrac{1}{11340}\right\}$ ft. $= \dfrac{610}{63}$ ft. $=$ 9 ft. 8 in....
22. 20 secs.
23. 2·8 secs.
24. 5 lbs. wt. upwards.
25. 4 lbs.
26. 64 tons.

XIV. b.

1. 4 sec. **2.** (i) 48 ft. (ii) 112 ft. **3.** 10 sec.; 1600 ft.
4. 64 velos; 64 ft. **5.** In 4 sec. from time stone is thrown.
6. 2 sec.; 192 ft. from the foot of the tower.

XV.

1. $\frac{16}{100}=\frac{4}{25}$ celos. **3.** Their ratio is 20 to 21. **4.** $\frac{16}{21}$ ft.
5. $\frac{64}{121}$ ft. **6.** $4\times 14\times 22$ poundals. **7.** 13 sec.
8. 770 lbs. weight. **9.** $(16800+770)=17570$ lbs. weight.
10. (i) 875 lbs. weight. (ii) 700 lbs. weight.
11. (i) 980 lbs. weight. (ii) is unaltered. **12.** $\frac{1}{35}$ celo.

XVI.

1. $300\times 20\times 112$ poundals.

2. $250\times 20\times 112$ poundals; $200\times 20\times 112$ poundals; $150\times 20\times 112$ poundals; $100\times 20\times 112$ poundals; $50\times 20\times 112$ poundals.

3. $500\times 20\times 112$ poundals.

4. (i) $\dfrac{220\times 20\times 112}{3}$ poundals. (ii) $44\times 20\times 112$ poundals. (iii) $\dfrac{22\times 20\times 112}{3}$ poundals. (iv) 176 yards.

5. (i) $\dfrac{8a}{15}$ celos. (ii) $\dfrac{7ma}{15}$ poundals. (iii) $\dfrac{ma}{5}$ p. (iv) $\dfrac{ma}{15}$ p.

6. (i) $\dfrac{10a}{10+n}$ celos. (ii) $\dfrac{n-4}{10+n}p$ poundals.

XVII.

1. $\frac{255}{8}$ poundals; 2 celos. **2.** 120 poundals; 8 celos.
3. $\frac{575}{12}$ poundals; $\frac{4}{3}$ celos. **4.** $\frac{255}{2}$ poundals; 2 celos.
6. $\frac{1}{2}$ a second. **7.** $30\frac{35}{66}$ ozs. **8.** $15\frac{1}{2}$ ozs. **9.** $12\frac{2}{29}$ lbs.
10. 17 : 15. **12.** $\dfrac{f-f'}{m+m'}$ celos.
14. Velocity before : velocity after $=\sqrt{3}:\sqrt{2}$.
15. 22×112 poundals. **16.** The weight of $1\frac{1}{2}$ cwt. **18.** 3600 ft.
19. $\frac{63}{64}$ of the weight of 2 ozs. **20.** $\frac{127}{128}$ of the weight of 2 ozs.
21. The lift has 2 celos upwards. **22.** 40 velos. **23.** 12 cwt.
24. $10\frac{20}{31}$ lbs. **25.** 16 secs. **27.** Since $4mm'$ is identically equal to $(m+m')^2-(m-m')^2$, $\therefore$ when $(m+m')$ is constant mm' diminishes when $(m-m')$ increases.

XVIII. a.

1. $44 \times 100 \times 2240$ pulses. **2.** $\frac{1}{3} \times 44 \times 2240$ poundals $= 1026\frac{2}{3}$ lbs. wt. **3.** 3 hrs. $3\frac{1}{3}'$. **4.** $264 \times 10000 \times 224$ pulses; $27\frac{1}{2}$ tons weight. **5.** 30 times $27\frac{1}{2}$ tons wt. $= 825$ tons wt. **6.** $22 \times 8000 \times 2240$ pulses; $18\frac{1}{3}$ tons wt. **7.** $88 \times 50 \times 2240$ pulses; 275 tons wt. **8.** 10 pulses; (i) 10 oz. wt. (ii) $3\frac{1}{8}$ lbs. wt. (iii) $312\frac{1}{2}$ lbs. wt. **9.** $28 \times 32 \times 15$ pulses. **10.** 88 strokes. **11.** $11\frac{1}{20}$ lbs. wt.

XVIII. b.

1. 896×2000 pulses; the weight of 1250 tons. **2.** $\frac{1}{60}$ sec.
3. 1600×1800 pulses; about $803\frac{4}{7}$ tons weight. **4.** $\frac{1}{20}$ sec.
5. 32 pulses; 10 lbs. weight. **6.** 55 pulses; $17\frac{3}{16}$ lbs. weight.
7. $66 \times 20 \times 112$ pulses; weight of $2\frac{1}{16}$ tons.
8. Force = weight of $10\frac{5}{7}$ cwt.; interval $= \frac{1}{480}$ second.
9. Force = weight of $2\frac{1}{2}$ cwt. + weight of hammer (14 lbs.). Distance $= 1\frac{5}{16}$ ft. **10.** (i) a weight of 160 tons + wt. of hammer (10 tons). (ii) 3 in. **11.** Force = weight of 8 tons + wt. of $\frac{1}{2}$ ton; interval $= \frac{1}{16}$ second.

XIX.

1. 10 velos. **2.** $\dfrac{mv}{m+m'}$ velos. **3.** 9 velos. **4.** $\frac{8}{15}$ velos.
5. $5\frac{1}{2}$ velos. **6.** $22\frac{1}{2}$ velos. **9.** 5 velos. **10.** $\frac{44}{9}$ velos.
11. $\frac{1200}{17921}$ velos. **12.** $(\sqrt{6}+\sqrt{2})$ sec. $= 3{\cdot}8636$ sec.
13. The retarding force is $k \times 2240 \times 32$ poundals; $\therefore$ the acceleration is $\dfrac{-k \times 2240 \times 32}{70 \times 2240} = -\dfrac{32k}{70}$ celos. The velocity after the impact is $\dfrac{660n - 32k}{70n}$ velos; $\therefore$ the interval in coming to rest is

$$\frac{660n - 32k}{70n} \div \frac{32k}{70} \text{ secs.} = \left(\frac{660}{32k} - \frac{1}{n}\right) \text{ secs.,}$$

or from the beginning of the impact $\dfrac{660}{32k}$ secs.

XX.

1. $15\frac{1}{5}$ velos; $19\frac{1}{5}$ velos. **2.** $-2\frac{2}{3}$ velos; $5\frac{1}{3}$ velos.
3. $-\frac{1}{2}v$; $5\frac{1}{2}v$. **4.** 8 velos.
6. $\left(\dfrac{e+1}{2}\right)v$; $\left(\dfrac{e+1}{2}\right)^2 v$; $\left(\dfrac{e+1}{2}\right)^3 v$; $\left(\dfrac{e+1}{2}\right)^n v$.
7. In 1 sec.; 16 ft. below the point from which m' is let fall;

$$v = -\frac{32}{m+m'}\{m' + 2em' - m\}; \quad v' = \frac{32}{m+m'}\{m + 2em - m'\},$$

v and v' being measured upwards.

8. $\frac{1}{1+2e}$. **9.** $7\frac{1}{2}$ velos in opposite direction to shot.

11. The impulsive tension causes a stress of opposite *sign* to that caused by an impact, hence we have $v+v'=u$ and $v-v'=-eu$, whence $v=\frac{1}{2}u(1-e)$; $v'=\frac{1}{2}u(1+e)$; after the first impact, the velocities are $\frac{1}{2}u(1+ee')$; $\frac{1}{2}u(1-ee')$.

12. $\sqrt{\frac{pa}{2m}}$ velos; $\sqrt{\frac{2pa}{3m}}$ velos; $\sqrt{\frac{n-1}{n}\cdot\frac{pa}{m}}$ velos.

XXI.

1. 1 second. **2.** $\sqrt{\frac{2}{3}}$.

3. (i) e^2h ft. (ii) $\frac{2e\sqrt{2h}}{\sqrt{g}}$ secs. (iii) $\left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}$ secs.

4. $\frac{2h}{1-e^2}-h$. **6.** 15·34... velos.

XXII.

1. $\frac{1}{20}$ velo; $\frac{1}{1200}$ celo. **2.** $\frac{22}{15}$ velo; $\frac{11}{27000}$ celo.

3. $\frac{3a}{b}$ velos; $\frac{3a}{b^2}$ celos. **4.** $\frac{3}{88}$ sec. **5.** 25 ft.

6. $32\sqrt{330}$ velos. **7.** $\frac{16000}{11}$ sec.; $(\frac{40}{11})^2 \times 400$ feet. **8.** 50 ft.

9. 32 lbs. **10.** $\frac{20 \times 112 \times 5280}{(60)^4}$ poundals. **11.** $\frac{32}{3{\cdot}28}$.

13. $66\frac{2}{3}$ ft. **15.** 4·54 secs. **18.** $\frac{fn}{m^2}$ celos.

21. (i) ·0328... velos. (ii) ·0328 celos.
(iii) (·0328...) × (·0022...) poundals.

XXIII.

1. 500 yards. **2.** 5 miles. **3.** 10·2 ft. nearly. **4.** 25 yards.
5. 103·6 ft. **6.** 14 feet; 2 feet; 10 feet; 13 feet nearly.
7. 350 yards; 60^0 with first direction.
8. 115 yds. nearly; making an angle $\tan^{-1}\frac{\sqrt{2}}{10+\sqrt{2}}$ East of North.
9. $20\sqrt{3}$ yards.
10. 3·6 miles, in a direction making the angle $\tan^{-1}\frac{1}{6}$ with the North.
11. 4·836 miles; $\cos^{-1}{\cdot}97849 = 11^0\ 54'$ [by the Tables].
12. $\frac{1}{6}\sqrt{\{(32)^2+(19)^2\}}$ miles $=$ 7·44... miles.

XXIV.

1. 50 yards; 86·5... yards.   **2.** 15 yards, 20 yards.
3. 323·8 yards.   **4.** $\sqrt{\{(662\cdot13)^2+(212\cdot13)^2\}}$.
5. ·707... mile; 0 miles.   **6.** 180 yards.

XXV.

1. 150 feet from its initial position; in a direction making an angle whose sine is $\frac{4}{5}$ to the East of North.
2. $\sqrt{\{(30)^2+(45)^2\}}$ feet from its initial position.
3. 24 ft. south of its initial position.
4. $8\sqrt{\{(35)^2+(15)^2\}}$ feet from its initial position.
5. $\sqrt{\{(200)^2+(150)^2\}}$ feet from its initial position.
6. The point is at rest.
7. $\sqrt{\{(40)^2+(64)^2\}}$ feet from its initial position.
8. 10 feet to the north of its initial position.

XXVI. a.

1. (i) horizontal distance 100 ft., vertical distance 16 ft. below the point of projection; (ii) horizontal distance 200 ft., vertical distance 64 ft. below the point of projection; (iii) horizontal distance 300 ft., vertical distance 144 ft. below the point of projection.
2. Draw OQ at an angle of 45^0 to the horizon, and then from Q draw QP vertically downwards so that (i) $OQ=100$ ft., $QP=16$ ft.; (ii) $OQ=200$ ft., $QP=64$ ft.; (iii) $OQ=300$ ft., $QP=144$ ft.
3. Draw OQ at an angle 30^0 to the horizon and then from Q draw QP vertically downwards so that (i) $OQ=1000$ ft., $QP=16$ ft.; (ii) $OQ=2000$ ft., $QP=64$ ft.; (iii) $OQ=3000$ ft., $QP=144$ ft.
4. 960 velos.   **5.** 225 ft. below the top.
6. $\sqrt{\{(16)^2+(11)^2\}}$ feet = about 19·4 feet.

XXVI. b.

	Range.	Time of flight.	Greatest Height.
1.	312·5 feet.	4·42... secs.	78·125 feet.
2.	78·125 feet.	2·21... secs.	19·53125 feet.
3.	128 feet.	2·828... secs.	32 feet.
4.	1691·4 feet.	7·8 secs.	244 feet.
5.	270·6 feet.	5·4 secs.	117·2 feet.
6.	67·6 feet.	2·7 secs.	29·3 feet.
7.	168·75 feet.	2·8125 secs.	31·640625 feet.

	Range.	Time of flight.	Greatest height.
8.	468·75 feet.	$6\frac{1}{4}$ secs.	156·25 feet.
9.	625 feet nearly.	·625 secs.	1·5625 feet.

10. 300 yards nearly. **11.** 1184 feet.
12. 1·88 feet and 2·28 feet. **13.** 9140 yards. **14.** 18490 yards.

XXVII.

1. 20 velos. **2.** 25 velos. **3.** 50 miles per hour $=73\frac{1}{3}$ velos. **4.** 8·2... velos.

5. $\sqrt{a^2+b^2}$ velos. **6.** $\sqrt{a^2+\left(\frac{22m}{15}\right)^2}$ velos. **7.** 3 velos.

8. $a\sqrt{3}$ velos. **9.** 6·7... velos. **10.** $10\sqrt{7}=26{\cdot}4$...velos.

11. 20 velos. **12.** 42·7 velos. **13.** 57·46 ft.; see Ex. i. p. 84.

14. 35·35... velos, 35·35... velos. **15.** 99·6 velos.

16. 1136 velos nearly. **17.** 28·28...velos. **18.** 120^0 from N.

19. It has received $(2-\sqrt{2})^{\frac{1}{2}}$ times its former velocity, making an angle $\cos^{-1}\frac{1}{2}(2-\sqrt{2})^{\frac{1}{2}}$ with it. **20.** 67 velos nearly. **21.** 30 velos.

22. 20·5 velos making angle with horizon $\tan=\frac{9}{5}$.

23. At a point $\sqrt{(20)^2+(\frac{88}{3})^2}$ ft. from point of stoppage.

24. 21 velos nearly. **25.** $8\sqrt{2}$ miles per hour.

26. At an angle with the vertical whose tangent is $\frac{1}{642816}$ which by the tables is about $\frac{1}{3}''$. **27.** About 23 velos.

28. About $(40-23)$ or 17 velos.

XXVIII.

1. Angle in direction of train whose $\tan=-\frac{3}{4}$. **2.** 13 celos.

3. 13 velos. **4.** 50·4... velos. **5.** 44·78 velos nearly.

6. 135^0 with direction of train.

XXIX. a.

1. 1 celo; 6 secs. **2.** 4·4 secs. **3.** It will be at a distance $20\sqrt{2}$ ft. N. W. of its initial position, and it will have $8\sqrt{2}$ velos.

4. The direction of motion makes an angle whose sine is $\frac{3}{5}$ with the force of 4 cwt. weight; the acceleration is 8 celos.

5. $\frac{3}{20}$ of 32 celos $=4{\cdot}8$ celos. **6.** The acceleration is $\frac{1}{2240}$ of $\sqrt{\{(180+80\sqrt{2})^2+(100\sqrt{3}+80\sqrt{2})^2\}}$ at a point distant about 1·64 ft. westwards and 1·6 ft. northwards of the starting point.

7. $\frac{1}{4}\sqrt{(14+9\sqrt{2})}$ celos $=1{\cdot}29$... celos.

8. 5·746... celos nearly [see Ex. i. p. 84]. **9.** ·6464... celos.

10. The sum of the resolved parts perpendicular to the plane of the forces acting on the mass [which is $(70\sqrt{3}+20)$ lbs. weight] must be less than the weight of the mass.

XXIX. b.

1. 2 secs. **2.** A force equal to the weight of $50\sqrt{2}$ lbs.

3. The acceleration is $\frac{3}{8}$ celos ; $\therefore$ the resultant force acting is 56×3 poundals ; $\therefore$ the force of friction $= (56 \times 32 - 56 \times 3)$ poundals $= 56 \times 29$ poundals.

5. The force acting $= 56 \times 32$ poundals $- \frac{1}{10}\, 56 \times \sqrt{3} \times 32$ poundals ; $\therefore$ the acceleration is $(16 - 1{\cdot}6 \times \sqrt{3})$ celos $= 13{\cdot}2288$ celos.

6. The weight of 1 ton. **7.** (i) the weight of $2\frac{7}{12}$ tons. (ii) the weight of $6\frac{7}{12}$ tons. **8.** 3 secs.; $\dfrac{108}{m}$ feet. **11.** $2 : 1$.

XXX.

2. $10\sqrt{(330)}$ seconds. **3.** $32\sqrt{(\frac{33}{10})}$ velos.

4. The whole time of flight $= \left\{\sqrt{\left(\dfrac{2l}{g \sin \alpha}\right)} + t_1\right\}$ secs. when t_1 is the positive root of the equation

$$h = \sqrt{(2gl \sin \alpha)}\, t \sin \alpha + \tfrac{1}{2} g t^2,$$

the horizontal distance from lower end $= \sqrt{(2g \sin \alpha l)}\, t_1 \cos \alpha$ ft. **6.** 30^0.

7. Through the fixed point draw a horizontal straight line; describe a circle to touch this line at the fixed point and also to touch the fixed line. The line joining the two points of contact is the required direction.

9. The construction is the same as that in Example **7**.

XXXI.

1. After $\frac{15}{32}$ seconds.

3. (i) After $\{12 \pm \sqrt{119}\}$ seconds, (ii) at an angle with the horizon whose tangent is $\pm \dfrac{\sqrt{119}}{5}$.

XXXII.

1. The axis of the parabola is the vertical line passing through the point at which the ball was dropped. The directrix is the horizontal line $\dfrac{66 \times 66}{2g}$ feet above this point.

2. $\left.\begin{matrix} 128\sqrt{3} \text{ ft.} \\ 64\sqrt{3} \text{ velos} \end{matrix}\right\}$ or $\left.\begin{matrix} 64\sqrt{3} \text{ ft.} \\ 64\sqrt{3} \text{ velos} \end{matrix}\right\}$ according as the angle 30^0 is one of elevation or depression. **4.** With $40\sqrt{6}$ velos. **5.** He cannot do it.

6. The least velocity is $u \cos \alpha$, which it attains after $\dfrac{u \sin \alpha}{g}$ seconds,

at a point whose vertical distance is $\dfrac{u^2 \sin^2 \alpha}{2g}$ ft. and whose horizontal distance is $\dfrac{u^2 \sin 2\alpha}{2g}$ ft.

10. 15^0.

12. $\frac{1}{2}$ second.

13. $12\sqrt{3}$ feet.

14. The time t_1 is the positive root of the equation

$$gt^2 + 2\sqrt{(2ga \sin \alpha)} \sin \alpha t = 2h,$$

and the distance $BC = \sqrt{\{h^2 + 2ga \sin \alpha t_1^2 \cos^2 \alpha\}}$ ft.

15. 44 feet.

16. 11 ft. 11 in.

17. $\frac{1}{4}\sqrt{15313}$ velos.

18. $u \cos \alpha$ velos.

19. $u \cos \alpha \sec \beta$ velos.

21. $\frac{1}{2}$ foot.

XXXIII.

1. $(1+e)\ 10$ velos.

2. They must interchange velocities in the line of impact and therefore [Ex. ii. p. 59] they must be (of equal mass and) perfectly elastic.

3. Their velocities are equal and each $= u\sqrt{\{\frac{1}{2}(1+e^2)\}}$.

4. $\sqrt{\{u^2 + \frac{1}{4}(1+e)^2 u'^2\}}$ and $\frac{1}{2}(1-e)\, u'$.

5. (i) 100 velos. (ii) 5 seconds.

6. (i) 50 velos. (ii) 2·8... seconds [Ex. iii. p. 40].

XXXIV.

1. 32 feet.

2. 1080 feet.

3. 91·6 velos if down the plane, $20\sqrt{\frac{13}{3}}$ if going up.

6. Let the angle of projection with the horizon be α, then the time of flight from the horizontal motion is $\dfrac{a}{u \cos \alpha} + \dfrac{2a}{eu \cos \alpha} + \dfrac{a}{e^2 u \cos \alpha}$ and from the vertical motion the time is $\dfrac{2u \sin \alpha}{g}$. Therefore

$$u^2 = \frac{ag}{\sin 2\alpha}\left\{1 + \frac{2}{e} + \frac{1}{e^2}\right\}.$$

7. $\cot \alpha = 2$.

9. The path of the ball is always parallel to a diagonal and it always returns to the point of projection.

13. $2eh \sin 2\alpha$ feet.

14. $\sin 2a = \dfrac{ag}{eu^2}$.

15. [See Ex. 6.] From the horizontal motion the time of flight $t_1 = \dfrac{b}{u \cos a}\left\{1 + \dfrac{2}{e} + \dfrac{1}{e^2}\right\}$. The time t_2 of going up to the ceiling is the least root of the equation $h = u \sin \alpha t - \frac{1}{2}gt^2$. The time of coming down t_3 is the positive root of the equation $h = evt + \frac{1}{2}gt^2$, where

$$v^2 = u^2 \sin^2 \alpha - 2gh;\ \text{also}\ t_1 = t_2 + t_3.$$

XXXV.

1. 20 celos. **2.** 80 poundals = weight of $2\frac{1}{2}$ lbs.

3. $\dfrac{mv^2}{p}$ feet. **6.** The weight of $1026\frac{2}{3}$ lbs.

7. The weight of $22\frac{44}{75}$ lbs.

8. (i) When at rest the tension is 56 lbs. weight.

(ii) When in motion the tension is $(56+70)$ lbs. weight.

9. The difference is about $94\frac{1}{6}$ lbs. weight.

10. At the distance of 3 feet from the 4 lbs.

11. 8 velos. **12.** $\frac{59}{32}$ lbs.

XXXIX.

1. $40 \times 56 \times g$ foot-poundals; 2240 foot-pounds.

2. $10 \times 14 \times 500 \times g$ foot-poundals; 70000 foot-pounds.

3. $10 \times 2240 \times 500 \times g$ foot-poundals; 11200000 foot-pounds.

4. $-12 \times 14 \times 200 \times g$ foot-poundals; -33600 foot-pounds.

5. $300 \times 2240 \times \frac{1}{80} \times 1760 \times 3 \times g$ foot-poundals; 70963200 foot-pounds.

6. $224 \times 600 \times g$ foot-poundals; 134400 foot-pounds.

7. 9 : 880. **8.** 22400 foot-pounds. **9.** $234\frac{2}{3}$ foot-pounds.

10. 50 velos $= 34\frac{1}{11}$ miles per hour. **11.** $224 \times 8 \times 88$ foot-pounds.

12. $19\frac{1}{6}$ miles per hour.

13. $\dfrac{96}{4+\cos\alpha}$ miles per hour, where $\cos\alpha = \sqrt{\{1-(\frac{1}{80})^2\}} = \cdot 9998\ldots$

that is, $19\frac{1}{6}$ miles per hour, very nearly. **14.** 29568 foot-pounds.

15. 5 : 2.

XL.

1. 80000 foot-poundals. **2.** $(2000)^2 \times 4 \times 112$ foot-poundals.

3. $(1600)^2 \times 6 \times 112$ foot-poundals.

4. $(88)^2 \times 200 \times 2240$ foot-poundals.

5. $\frac{1}{25} \times (88)^2 \times 84$ foot-poundals $= 813\cdot 12$ foot-pounds.

6. 1680 foot-pounds.

7. $\{(66)^2 \times 180 \times 2240 + 3 \times 1120 \times 1760 \times 6 \times 32\}$ foot-poundals

$= 2240 \times 9 \times 11 \{7920 + 5120\}$ foot-poundals.

8. $\{(88)^2 \times 50 \times 2240 + 500 \times 100 \times 2240 \times 32\}$ foot-poundals

$= 2240 \times 64 \times 50 \{121 + 500\}$ foot-poundals.

9. The necessary work is $112 \times 8 \times 32$ foot-poundals; the superfluous work is $112 \times \frac{1}{2} \times 25$ foot-poundals. Their ratio is 512 : 25.

10. No work is done while the brakes are applied, that is, for 110 yards. The work done in producing kinetic energy is

$\frac{1}{2}(11)^2 \times 2240 \times 10$ foot-poundals, the work done against friction is $28 \times 32 \times 1650 \times 3$ foot-poundals. The total work done is

$$2240 \times 11\,\{55+180\} = 2240 \times 11 \times 235 \text{ foot-poundals.}$$

The work done in going a mile with uniform velocity

$$= 28 \times 32 \times 1760 \times 3 = 2240 \times 11 \times 192 \text{ foot-poundals.}$$

These two are in the ratio $235 : 192$.

11. Let the train be m lbs. The work done in the first case is $\{\frac{1}{2}(66)^2 m + \frac{32}{120} \times m \times 1760 \times 3\}$ foot-poundals, in the second

$$\frac{32}{120} \times m \times 1760 \times 3 \text{ foot-poundals.}$$

Their ratio is $3586 : 1408 = 163 : 64$.

12. $11 \times 2240\{55-12\}$ foot-poundals.

XLI.

2. 80 velos. **7.** See Art. 133.

XLII.

1. $716\frac{4}{5}$ H. P. **2.** 30 miles per hour. **3.** 1·564...
4. $31\frac{1}{4}$ miles per hour. **5.** $93\frac{3}{4}$ lbs. weight. **6.** $93\frac{3}{4}$ tons.
7. It does work at the rate of $6\cdot0\dot{6}$ H. P. against gravity; it also delivers $\frac{50}{3}$ gallons of water per second with $\frac{50}{3}$ velos; hence it must produce also work at the rate of ($\frac{1}{2}mv^2$, i.e. of) $\frac{1}{2} \times \frac{500}{3} \times (\frac{50}{3})^2 \times \frac{1}{32} \times \frac{1}{550}$ H.P.; that is about 7·2... H. P. **8.** $\cdot 101\dot{8}$ H. P.
9. 750 sq. in. [The pressure of the steam is applied to each side of the piston in turn.] **10.** 67·4... lbs. **11.** 7·5. **12.** 9 bus.

MISCELLANEOUS PROBLEMS. XLIII.

1. $\frac{1}{3}$ celo. **2.** $\frac{5}{3}$ celo. **3.** $-7\frac{1}{3}$ velos, $51\frac{1}{3}$ velos. **4.** The weight of 35 cwt.; $2\frac{1}{2}$ minutes. **5.** 492 grammes. **6.** The weight of 84 lbs. at the surface of the earth. **8.** Weight of 1 cwt.; $\frac{2}{5}$ celos. **10.** $114\frac{2}{3}$ velos. **11.** 9 min. 10 secs. **12.** Weight of $8\frac{2}{3}$ cwt. **13.** 144 tons. **14.** Weight of $184\frac{1}{3}$ lbs. **15.** Weight of $(184\frac{1}{3} + 32\frac{1}{12})$ lbs. **16.** $19\frac{7}{16}$ tons. **17.** Weight of 30 oz. upwards. **18.** $\frac{40}{61}$ lbs. **19.** $5\frac{1}{3}$ celos downwards. **20.** $2\frac{2}{3}$ celos upwards. **21.** $1\frac{3}{17}$ lbs. **22.** 15 ft. $1\frac{1}{2}$ in. **23.** About 80 velos. **24.** $15°$ west of South; $2\sqrt{3}$ miles per hour.

25. Weight of $\frac{n}{m}$ tons; $\frac{448}{75}\frac{nv}{m}$ additional horse power. **26.** 80 feet. **27.** 50 tons. **29.** $13\sqrt{\frac{2(m+n)l}{m-n}}$ seconds; weight of $\frac{1}{13} \times \frac{\cdot 2mn}{m+n}$ tons. **30.** $\frac{2}{63}n$ oz. **37.** The unit force $=\frac{1}{32} \times \frac{1}{16} \times \frac{1}{12}$ of the weight of 1 lb. **38.** $\frac{8}{3}\sqrt{3}$ velos. **39.** Weight of $\frac{175}{96}$ oz.; about twice the weight of the projectile. **40.** Eleven times its own weight. **41.** The velocity of projection is to be as small as possible; $\therefore$ the directrix of the parabola must be as low as possible. Let *AB* be the points on the tower through which the projectile passes; then the focus of the parabola must bisect *AB*. **42.** $1\frac{1}{8}$ lbs. The *energy* varies as the amount of powder. **44.** $112\frac{1}{2}$ feet; $3\frac{3}{4}$ secs. **45.** $4\sqrt{\frac{11}{6}}$ velos. **46.** Weight of 600 lbs. **47.** Weight of $12\frac{1}{2}$ cwt. **52.** $\frac{11}{120}$. **75.** $W : P = 4n^2 - ng : g - 4$. **78.** $56\frac{1}{4}$ miles per hour and $28\frac{1}{8}$ miles per hour. **79.** $\sqrt{(13-6\sqrt{3})}$ miles per hour. **114.** 40 feet from the point in the ground vertically below the centre of the circle. **115.** Their velocities are as $1 : \sqrt{5}$. **116.** $\frac{8}{3}\sqrt{3}$ lbs. weight; $8\sqrt{\frac{2}{\sqrt{3}}}$ velos. **128.** $3v$ and v when at the same point and when at the opposite ends of a diameter respectively. **129.** Let the radius to the point make the angle α with the horizon, then $\sin\alpha = \frac{gr}{v^2}$, where r = the radius of the wheel and v the velocity of the coach.

130. $(3+\frac{3}{2}\sqrt{3}) = 5\cdot 6$ feet; $6+5\sqrt{3} = 14\cdot 66$ ft.

131. $\frac{100 \times 440 \times 3 \times 20 \times 11 \times 1024}{16 \times 20 \times 112}$ foot-tons per second. **132.** A plane having its line of greatest slope parallel to that of the inclined plane. **209.** A mass-acceleration multiplied by a velocity.

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